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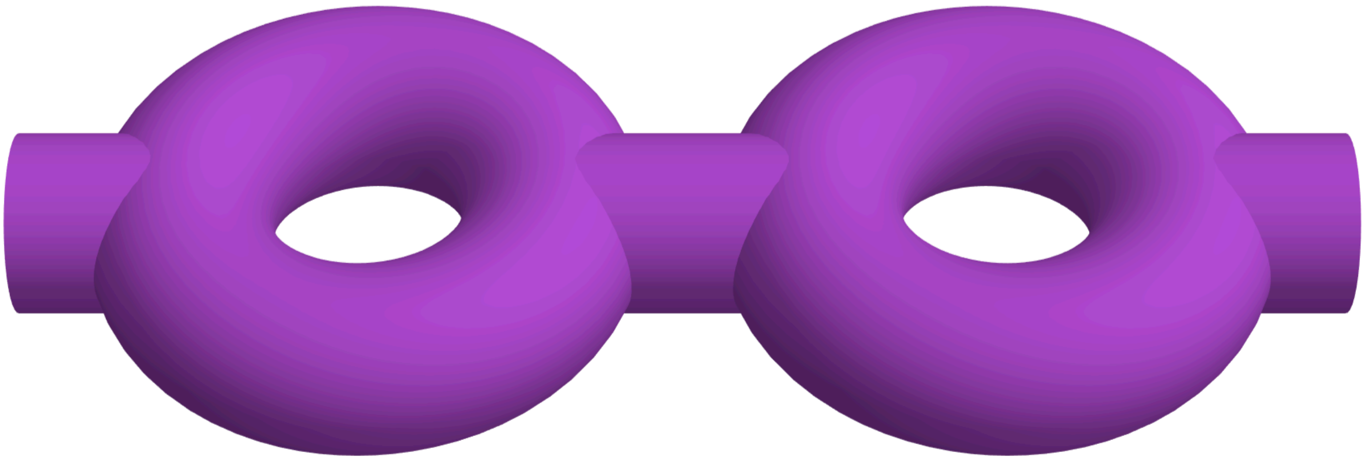


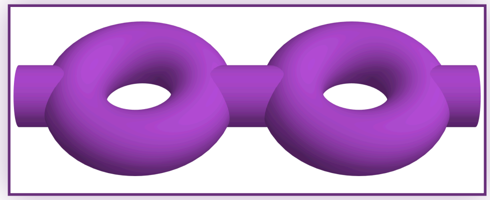
Inter-universal Teichmüller Theory
Conversations with Mochizuki

James Douglas Boyd

SciSci Research, Inc.

About the Cover:

A stylized illustration of Figure I2.1 from the first IUT paper, “Inter–universal Teichmüller Theory I: Construction of Hodge Theaters”, which shows two Hodge theaters opposite a theta link.



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Contents

1 Introduction

1.1 Preface

The following piece is based on SciSci interviews with Professor MOCHIZUKI Shinichi (望月新一) conducted at the Research Institute for Mathematical Sciences (数理解析研究所), or RIMS, at Kyoto University (京都大学), during a month-long visit to RIMS, and in coordination with a *Centre national de la recherche scientifique* (CNRS; French National Centre for Scientific Research) project by the name of Arithmetic and Homotopic Galois Theory (AHGT).

The exchanges that Mochizuki-sensei (as one says in Japan) and I shared during my RIMS visit – transpiring over several weeks, and over a dozen hours of discussion – constitute his first-ever participation in an interview series. For this unprecedented opportunity, SciSci owes Mochizuki-sensei a debt of gratitude.

Mochizuki-sensei is an arithmetic geometer, perhaps best known internationally for his 1996 proof of Grothendieck's anabelian conjecture and his development of inter-universal Teichmüller theory (IUT), which has attracted public intrigue and media attention to a degree perhaps unrivaled by any other recently developed theory in arithmetic or Diophantine geometry.

SciSci came to Mochizuki-sensei with many technical questions about numerous theoretical aspects of IUT. Mochizuki-sensei answered all of the questions asked by SciSci, and, in many cases, was of the view that SciSci's questions presented an opportunity to articulate aspects of his vision for IUT that are not explicitly written in the IUT papers. In certain cases, our conversations also led to new perspectives on the theory itself. Mochizuki-sensei himself remarked –

Mochizuki-sensei:

> If someone doesn't write this down, it will be forgotten, or will never be even noticed by the rest of the world. I try to record as much as possible, in various ways, but – I think I've said so many things during our meetings that are not written anywhere. If you don't write them down – if you don't record them in some way – they will be lost to history.

The following piece does endeavor to delve into the technical depths of the IUT texts in some detail. However, the following is not a mathematical paper. Little formulation is given, though some symbols are used for convenience on occasion; the treatment is entirely conceptual. For instance, I'll write "a smooth, proper, hyperbolic curve over a number field" rather than use the notation for the curve or the field. Moreover, this is not a general pedagogical work; one won't find introductions to prerequisite topics such as scheme theory, Galois theory, or p-adic numbers. SciSci apologizes for the inconvenience; covering general prerequisites is infeasible. Nevertheless, in our present age, resources on such prerequisite topics are but an internet search away; one can catch up on prerequisites that way.

1.2 What is IUT? A Brief Summary

IUT is presented in a four-part series of papers, spanning 723 pages:

- "Inter-universal Teichmüller Theory I: Construction of Hodge Theaters"
- "Inter-universal Teichmüller Theory II: Hodge–Arakelov-Theoretic Evaluation"
- "Inter-universal Teichmüller Theory III: Canonical Splittings of the Log-Theta-Lattice"
- "Inter-universal Teichmüller Theory IV: Log-Volume Computations and Set-Theoretic Foundations"

These are now published in a special issue of *Publications of the Research Institute for Mathematical Sciences*, Volume 57, No. 1/2 (2021) – distributed by the European Mathematical Society.

Years back, when I first began my endeavor to study IUT, I vied expeditiously to find a 2-4 paragraph survey, necessarily outside of an academic paper. Before even seeing the notation and reading the texts (or even expositions) in full, I wanted to know what ideas the notation captured, and what the texts communicated. Here, I'll attempt to provide something comparable; the kind of short summary that I had sought. Included below is an attempt at a high-level, 4-paragraph summary of IUT, which might be accessible to someone with prior exposure to certain topics in undergraduate-level mathematics (e.g., complex analysis, abstract algebra, arithmetic geometry, class field theory).

The first summary remark to be given on IUT is that it is concerned with the entanglement of the multiplicative and additive dimensions of ring structure. We can review how it addresses this question by beginning with the Teichmüller-theoretic nature of IUT. Unlike complex-analytic Teichmüller theory or p-adic Teichmüller theory (the latter also developed by Mochizuki-sensei), inter-universal Teichmüller theory is concerned with deformations and rotations of what is termed arithmetic holomorphic structure. By way of a cursory preview, we can, for now, introduce arithmetic holomorphic structure as a treatment of the multiplicative ($\boxtimes$) and additive ($\boxplus$) dimensions of a ring in a manner analogous to the real and imaginary dimensions of $\mathbb{C}$ in complex analysis. Arithmetic holomorphic structure refers to more than $\boxtimes$/$\boxplus$-structure, however; it refers to the rigid coupling between such dimensions. Teichmüller deformations are performed in order to examine the effects of a change in $\boxtimes$-structure (i.e., dilations) on $\boxplus$-structure. So, one inquires into the very structure of ring theory via assessment of $\boxplus$-structure responses under $\boxtimes$-deformation. Such dilations are comparable to quasi-conformal mappings in complex-analytic Teichmüller theory, which alter real data relative to their imaginary counterpart. Thus, part of understanding IUT is grasping the manner in which it abstracts the properties of Teichmüller theory. IUT, classical complex-analytic Teichmüller theory, and p-adic Teichmüller theory share what we might call a certain common "Teichmüller-theoreticity", manifested in the case of IUT as a Teichmüller theory for arithmetic holomorphic structure.

In IUT, developing a Teichmüller theory for arithmetic holomorphic structure leads naturally to the construction of what is called the log-theta-lattice. The arrows in the lattice correspond to gluings of Hodge theaters (which are models of ring/scheme theory). The construction of this lattice proceeds by first taking a non-ring/scheme-theoretic gluing known as the theta-link, which respects $\boxtimes$-structure but not $\boxplus$-structure, thus effectuating a deformation. (Indeed, in a Teichmüller theory for arithmetic holomorphic structure, this non-ring/scheme-theoretic gluing is a deformation.) After having deformed one's ring structure, one must then assess the effect on $\boxplus$-structure by measuring the extent to which one can recover the ring structure. (On a methodological note: recovery, or reconstruction, is a key practice in the *modus operandi* of anabelian geometry.) Such is achieved (up to mild indeterminacy) by making use of another gluing, known as the log-link, whose origination can indeed be found in p-adic absolute anabelian geometry. Theta- and log-links form the log-theta-lattice.

Following reconstruction, the distortion incurred by the theta-link (or, alternatively, the irrecoverability of the entanglement of $\boxtimes$/$\boxplus$-structure), is quantified with log-shells and log-volumes. Thus, ring structure is iteratively deformed and reconstructed in the log-theta-lattice, from which one computes an upper bound on the limitations of reconstructability. The domains and codomains of theta- and log- links in the lattice are Hodge theaters (again, models of ring/scheme theory), each with its own collection of Grothendieck universes; hence the moniker 'inter-universal'. One endeavors to recover the ring structure of one Hodge theater from the 'point of view' of another Hodge theater following a Teichmüller deformation. The log-theta-lattice is used to compute bounds on the irrecoverability of arithmetic holomorphic structure. Bounds on the limitations of recoverability carry crucial arithmetic information concerning the entanglement between addition and multiplication. If $\boxplus$-structure can be almost fully recovered, up to mild indeterminacies, in the domain of the ($\boxtimes$-preserving, $\boxplus$-violating) theta-link, then the multiplicative and additive combinatorial dimensions of ring structure are indeed entangled; one cannot separate one from the other, except to a mild degree.

The development of IUT proceeded, in part, with certain Diophantine inequalities (i.e., the Szpiro conjecture) borne in mind as motivating problems. Precedent might lead one to endeavor to attend to them through ascertaining bounds on heights of elliptic curves by way of determining invariance under isogeny. Previous attempts to do so by Mochizuki-sensei, via Hodge-Arakelov theory, were met with impediments due to the inadequacies of ring/scheme-theoretic strategies. The desirable theta function respects $\boxtimes$-structure, but not $\boxplus$-structure. *Prima facie*, such might be considered a methodological hindrance. However, alongside the Szpiro conjecture, one has the *abc* conjecture, another Diophantine inequality encoded solely in terms of, effectively, the 'entanglement between multiplication and addition'. Thus, one might hypothesize that the non-ring/scheme-theoretic mappings of Diophantine interest, rather than falling short of methodological desiderata due to violation of underlying ring structure, might in fact be repurposed to ascertain the nature of the entanglement between $\boxtimes$- and $\boxplus$-structure within a Diophantine setting, and thereby prove the Szpiro and *abc* conjectures. However, this would require a theory in which the violation and recovery of ring structure was made a method in its own right; IUT can be seen as a response thereto. Thus, in IUT, ring structures appear in Hodge theaters, which are situated in a log-theta-lattice for purposes of distortion and recovery. Although this approach is perhaps most famous for its Diophantine implications, *abc* is but a motivating problem encapsulating the greater structural relationship of the entanglement between multiplicative and additive dimensions, that is, the nature of arithmetic holomorphic structure, for which IUT, as the name suggests, provides a Teichmüller theory and inter-universal framework.

So, if you find yourself at a cocktail party (or indeed a similar venue of engagement) and the topic of IUT is raised, you could try to encapsulate $\sim$720 pages with a one-liner: *IUT is a Teichmüller theory for deforming and reconstructing ring structure, across universes, so as to elucidate the nature of the entanglement between multiplication and addition.*

The above is a portrait – one that is, admittedly, softened and smoothened in order to make a gentle impression – of the theoretical and methodological crux of IUT. Much of the mathematical inventory of concepts found in the IUT papers – either fashioned in precursor works or invented in the IUT papers – including:

- Prime-strips
- Frobenioids
- Multiradiality
- Coricity
- Log-shells and log-volumes
- Mono-analytic containers
- Pilot objects
- Belyi cuspidalizations
- Species
- Logarithmic Gaussian procession monoids
- Theta monoids
- Cyclotomic synchronizations
- Indeterminacies
- Canonical log-theta-lattice splittings
- Log-Kummer correspondences
- Mono-theta environments
- $\in$-loops

can be regarded as being introduced

and deployed in service of the theoretical and methodological crux of IUT summarized above. They are mostly involved in:

- The decomposition, deformation, and reconstruction of ring structure
- Quantification of the distortions that arise during reconstruction
- Handling primes in such a way as to give canonical Diophantine significance to a Teichmüller theory of arithmetic holomorphic structure
- Describing the algorithmic nature of the theory with respect to the foundations of ZFCG (Zermelo-Fraenkel + the Axiom of Choice + the existence of Grothendieck universes)

Thus, although the conceptual inventory of IUT is rather sizable, the role of each concept can, coarsely speaking, be assigned, in some way, to the above strategies (and their relations).

So, with these summary remarks on IUT in mind, let us now proceed to inquire into the details of the theory and its history.

2 A History of the Mathematics of IUT

2.1 From Hodge-Arakelov to Inter-universality

During preparations for my visit to Kyoto, whilst browsing the chronology of Mochizuki-sensei's lectures (which he maintains on his website) the 2000-2004 interval struck me as intriguing. Between 1999 and 2000, Mochizuki-sensei delivered several lectures on "The Hodge-Arakelov Theory of Elliptic Curves". In 2004, he lectured in Tokyo for the very first time on a decidedly new topic: "A Brief Introduction to Inter-universal Geometry (Part I)". Notably, no lectures are listed in the chronology as having been given between the 1999/2000 Hodge-Arakelov theory lectures and the 2004 lecture. Given the paradigmatically distinct character of inter-universal geometry relative to Hodge-Arakelov theory, it is evident that the 2000-2004 period was far from a mathematical hiatus or interregnum; much had transpired therein. Indeed, as Mochizuki-sensei confirmed, the 2000-2004 period is an appropriate one to examine.

Mochizuki-sensei:

> I finished working on Hodge-Arakelov theory [...] around 2000. Around 2001 and 2002, I came up with the term "inter-universal".

In hindsight, this neoteric development, "inter-universal geometry", can be seen as a kind of proof of concept or prototype of IUT, *avant la lettre*. The lecture was a preview of a theory to come.

Mochizuki-sensei:

> My general feeling is [...] I gave it at a time when [...] things were still in a very primitive state.

Thence, lectures were delivered on the "geometry of categories" and "Inter-universal Hodge-Arakelov theory". The ideas intimated therein eventually culminated in what we now call IUT.

In "§1.1 Motivation" of the inter-universal geometry lecture slides, the prospect of attaining a proof of the *abc* conjecture is mentioned as a problem which, concerning the relationship between multiplication and addition, one can bear in mind as one considers the impetus for inter-universal geometry. (*abc* is also mentioned, to be sure, in his 2000 Hodge-Arakelov theory lecture given at the "Algebraic Geometry 2000, Azumino" conference.) Moreover, we are told that scheme-theoretic Hodge-Arakelov theory, the previous approach developed by Mochizuki-sensei, is now seen to be impeded by a certain technical obstacle that might be resolved via, in essence, an inter-universal approach.

After arriving in Kyoto, seeking to understand the lines of thought and turning points that arose during the formative 2000-2004 period, I spent some time with Mochizuki-sensei at RIMS endeavoring to recall the early genealogy of IUT. We proceeded to trace the origins of the architectural mainstays of IUT, including the vestigiality of resources from prior theories, the uptake of new inventions, and the decommissioning of outdated concepts. Notwithstanding this being his first interview, and the conceptually demanding and historiographically novel nature of the endeavor, Mochizuki-sensei was willing to trace this genealogy, despite the requirement of re-engaging with subject matter far afield from his present work.

Mochizuki-sensei:

> My mind is somewhat focused on the research that I'm doing now, so I need to reconstruct what my state of mind was 20 or 30 years ago. [...] Many of these ideas were attempts that became obsolete. But still, there is a core portion of these ideas that did not become obsolete.

However, we quickly found our footing in threading highly interwoven conceptual developments along a multi-decadal and pan-theoretical arc, identifying mathematical developments whose communication Mochizuki-sensei felt would be of interest even to a mathematical audience.

Mochizuki-sensei:

> I think [this can] really help people to understand what IUT is about. [...] I think all of this would be of interest to people – there's this intertwining of history and mathematical content.

So, without standing on ceremony any longer, let's begin.

With the 2004 inter-universal geometry lecture commencing with the inadequacies of Hodge-Arakelov theory, it's appropriate enough to begin with an overview of Hodge-Arakelov. Our overview will be abetted by first introducing, as it goes, another theory: what is known as arithmetic Kodaira-Spencer theory. Actually, over the course of the development of both Hodge-Arakelov theory and IUT, arithmetic Kodaira-Spencer theory was more of a phantom presence, a specter of a theory to be realized. In fact, a strategic aim of Hodge-Arakelov theory can, with brevity, be distilled with reference to this specter: Hodge-Arakelov theory, and its Comparison Isomorphism, is pursued for purposes of realizing the arithmetic Kodaira-Spencer morphism and proving certain Diophantine inequalities therewith.

Prior to meeting with Mochizuki-sensei in Kyoto, I had happened upon §2.3, "Towards an Arithmetic Kodaira Spencer Theory", in the book *Foundations of p-adic Teichmüller Theory*, a 1999 joint publication of the American Mathematical Society (AMS) and International Press (IP). This section summarizes the strategy that Mochizuki-sensei developed, with Diophantine inequalities in mind. An attempt to execute such a strategy was indeed made with Hodge-Arakelov theory, prior to IUT. That the strategy is articulated in the p-adic Teichmüller theory volume is also worthy of note (and to be discussed in a subsequent section).

In §2.3, noting the profitable use of the Kodaira-Spencer morphism for function fields in the proof of the geometric version of Vojta's conjecture, Mochizuki-sensei asks whether an arithmetic analogue of the Kodaira-Spencer map could be fashioned to prove Vojta's conjecture over number fields. However, it is common knowledge in number theory that one cannot handily exchange function-field-theoretic advances for number-field-theoretic ones; function fields give one the extra berth to differentiate, which, alas, is unavailable in the number field case. Nonetheless, Mochizuki-sensei proceeded to draft the wishlist of arithmetic analogues which one would need. One would like some analogue of a constant curve over a number field. One wishes to have an absolute field of constants over which to differentiate. Several conditions would need to be satisfied to give embodiment to this specter of a theory.

Mochizuki-sensei:

> There are various aspects that one has to solve to get a number field analogue of the Kodaira-Spencer map.

The Kodaira-Spencer map can be synopsized morally as a correspondence between a) motion in the base-space of a family of elliptic curves, and b) deformation in the Hodge filtration of the de Rham cohomology of the elliptic curve induced by such a deformation. Hodge-Arakelov theory was successful in procuring many of the needed analogues for establishing an arithmetic version of this correspondence. In lieu of absolute constants, one uses d-torsion points (where d is a positive integer). Galois groups/fundamental group(oids) give the analogue of base-space motion. The Hodge filtration corresponds to the evaluation of theta functions and their derivatives at d-torsion points.

Kodaira–Spencer Morphism	Arithmetic Analogue
Motion in the base–space of a family of elliptic curves	Galois groups/fundamental group(oid)s
Deformation in the Hodge filtration of the de Rham cohomology of the elliptic curve obtained by such motion	Non–preservation of Hodge filtration from evaluation of theta functions and derivatives at d–torsion points

Figure 1: Comparison of Kodaira-Spencer theory and arithmetic Kodaira-Spencer theory

The resulting Comparison Isomorphism – analogous to the Comparison Isomorphism between de Rham and étale/singular cohomologies in complex and p-adic Hodge theories, now within a global-arithmetic (i.e., Arakelov) setting – is the prize of the theory.

Mochizuki-sensei:

> Hodge-Arakelov theory was meaningful because it allows one to prove this Comparison Isomorphism in the number field case; and then you can see how the Galois action affects the Hodge filtration, which is very much like the Kodaira-Spencer morphism.

Readers of the IUT papers will recall the title of the second paper in the *Publications of the Research Institute for Mathematical Sciences* (PRIMS) special issue, "Inter-universal Teichmüller Theory II: Hodge–Arakelov-Theoretic Evaluation", which pertains to a certain identification, obtained via a non-scheme-theoretic theta-link between distinct Hodge theaters. The theta-link is constructed by identifying the multiplicative monoid obtained by evaluating, at certain l-torsion points (where l is a prime number ≥ 5), the reciprocal of the l-th root of the theta function that arises from one Hodge theater at primes of bad reduction with the multiplicative monoid generated by the $2l$-th roots of the q-parameter associated to the other Hodge theater. The precursor to the functions in the domain and codomain of the theta-link is the theta function of Hodge–Arakelov theory, which is of dear importance in obtaining a Hodge filtration.

Mochizuki-sensei:

> In some sense, what Hodge-Arakelov theory is saying is: basically you're considering theta functions and their derivatives. [...] [For] the theta functions that correspond to a line bundle of degree d [. . .] – that function space – you can look at the values restricted to the d-torsion points, [which] gives an isomorphism. [...] If you just look at set-theoretic functions on the torsion points, you don't have a Hodge filtration, but if you think about these theta functions and their derivatives, that gives a filtration. You have the theta functions, then the first derivatives, and the second derivatives, and so on; and that gives a filtration. And you want to see how that filtration is affected

> by automorphisms of the module of d-torsion points.

Now, as is discussed unequivocally in the Hodge-Arakelov literature, the theory was impeded by the obstacle of Gaussian poles, which can be thought of as a manifestation of an undesirable Gaussian factor when mapping from the de-Rham-cohomological space of polynomials to the étale-cohomological space of set-theoretical functions.

Mochizuki-sensei:

> I was going in the right direction, but I didn't see precisely what to do. [...] There was something incomplete about Hodge-Arakelov theory. [...] Around 2000, 2001, [and] 2002, I was interested in getting rid of these Gaussian poles.

In IUT – or from the point of view of the framework of arithmetic holomorphic structure – Gaussian poles can be thought of as a sort of distant ancestor of the theta-link of IUT, which links distinct arithmetic holomorphic structures via a common mono-analytic structure. The multiradiality of the multiradial representation of the theta-pilot in IUT, involving tensor-packets of log-shells, may then be regarded as analogous to the crystalline nature of the crystalline theta object and its concrete function-theoretic representation via the theta trivialization (i.e., realizing the arithmetic Kodaira-Spencer morphism without Gaussian poles).

One particularly consequential attempt at amelioration of the Gaussian pole dilemma was the development of the scheme-theoretic theta convolution, which implements an arithmetic analogue of (the inverse of) a certain convolution from Fourier analysis (i.e., convolution with the theta function), where we recall that Gaussians correspond, essentially, to Fourier transforms of theta functions. This involved the development of an analogue of (finite) abelian-group-theoretic Fourier analysis for (finite, flat) commutative schemes. Although yielding still-dissatisfactory fruits, it was indeed consequential insofar as it later prompted Mochizuki-sensei to inquire into an alternative beyond such scheme-theoretic approaches, which was a crucial impetus for the theta-link of IUT.

Mochizuki-sensei:

> The theta convolution was very much an approach to solving this problem in Hodge-Arakelov theory in a very elementary, scheme-theoretic way. It gives something like the functional equation [of the theta function] at the level of torsion points. [...] Ultimately, I was led to the idea of the theta-link: data consisting of the theta values q^{j^2} [...] [being] isomorphic to q. Its multiplicative data is isomorphic – [but] it's not compatible with the additive structure.

Such is the non-scheme-theoretic nature of the theta-link; it respects multiplicative structure, but not additive structure, and therefore does not give a ring homomorphism.

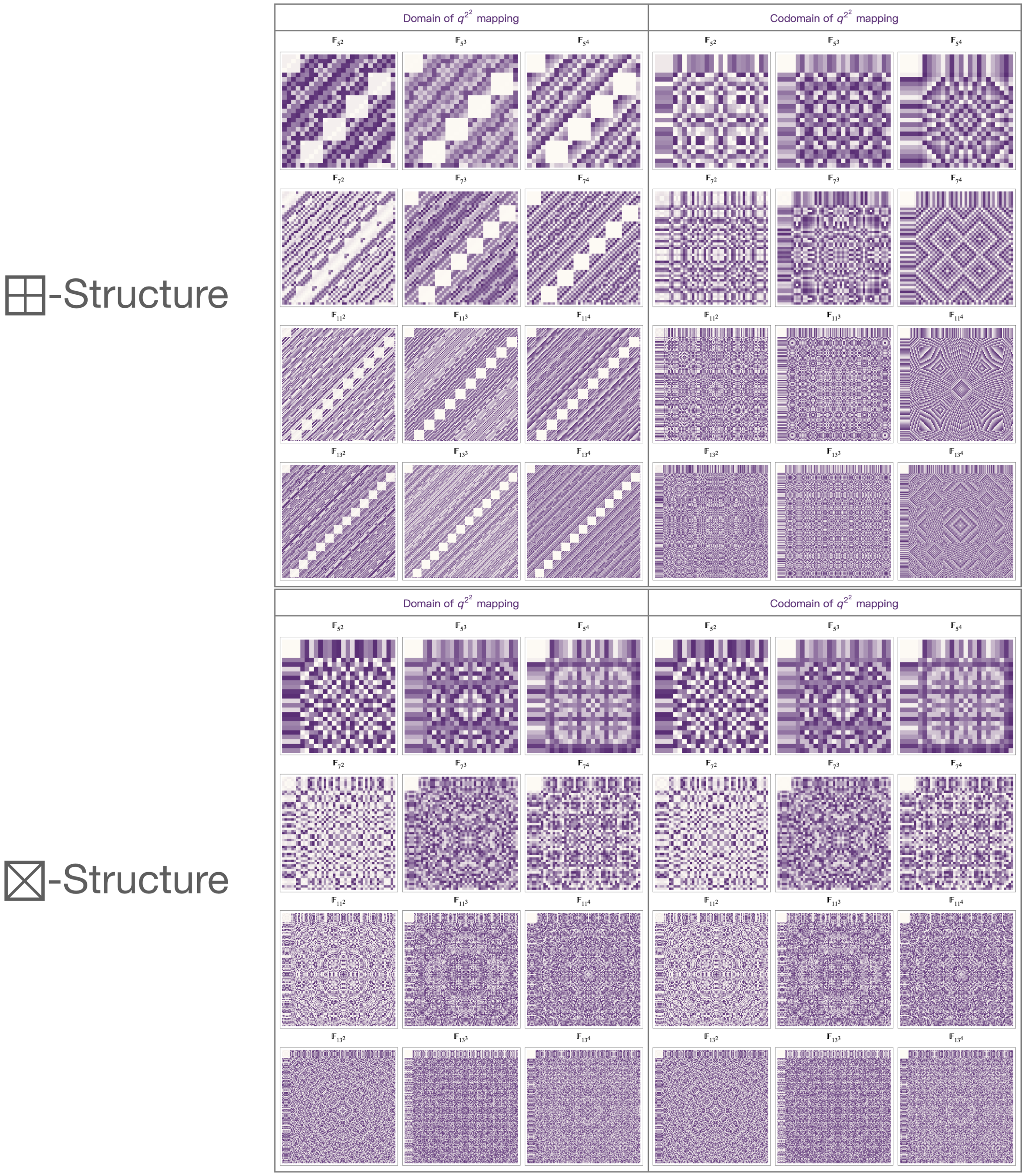

Figure 2: An elementary demonstration that mapping to q^{j^2} is non-scheme-theoretic (for the case of $j = 2$). Addition and multiplication tables for various finite fields, $\mathbb{F}_{p^n}$, are shown (where $p \in \{5, 7, 11, 13\}$, $n \in \{2, 3, 4\}$). Multiplicative structure is respected, but not additive structure.

The theta-link plays a role in IUT analogous to quasi-conformal mappings, or Teichmüller dilations, in complex-analytic Teichmüller theory. It can, to a large extent, be regarded as a harbinger of a decisive methodological pivot from scheme-theoretic Hodge-Arakelov theory to IUT, the latter being a Teichmüller theory for arithmetic holomorphic structure, or, a Teichmüller theory for deforming and rotating ring structure. However, IUT, although developed in part to surmount the limitations of scheme-theoretic Hodge-Arakelov theory, does not itself abandon scheme theory; rather, it subjects models of ring/scheme theory (plural) to a new Teichmüller theory, and does so across Hodge theaters in an inter-universal setting (whence the name inter-universal Teichmüller theory). This involves granting each Hodge theater, with its respective collection of Grothendieck universes, a distinct ring structure, mapping between Hodge theaters non-scheme-theoretically via the theta-link, and recovering the alien ring structure of one Hodge theater in another Hodge theater via the construction of invariants with respect to the log-link.

This particular aspect of the leap from Hodge-Arakelov theory to IUT can be summarized as follows. Noting the shortcomings of ring/scheme theory, or the manner in which the desired Diophantine-geometric approach violates scheme theory, one might choose not to discard ring/scheme theory itself, but instead to 'pluralize' models of ring/scheme theory in an inter-universal setting, thereby allowing one to quantify the structural breakdown of models of ring/scheme theory under Teichmüller deformation. This structural breakdown encodes information on the entanglement between additive and multiplicative structure. Here, we find a new way of looking at mathematical theories. Given that theories, ultimately, are regimes used to encode mathematical structure, circumstances under which they break down can help reveal the relations between the structures they encode. However, such revelation requires a greater setting in which one can acquire an external perspective on the theory under scrutiny. One need not remain sequestered within a theory; one can step outside of it into a wider setting, violate the theory, and examine the ways in which its constitutive structure can be repaired and the degree to which it cannot. Teichmüller-theoreticity and inter-universality are introduced to furnish such an approach.

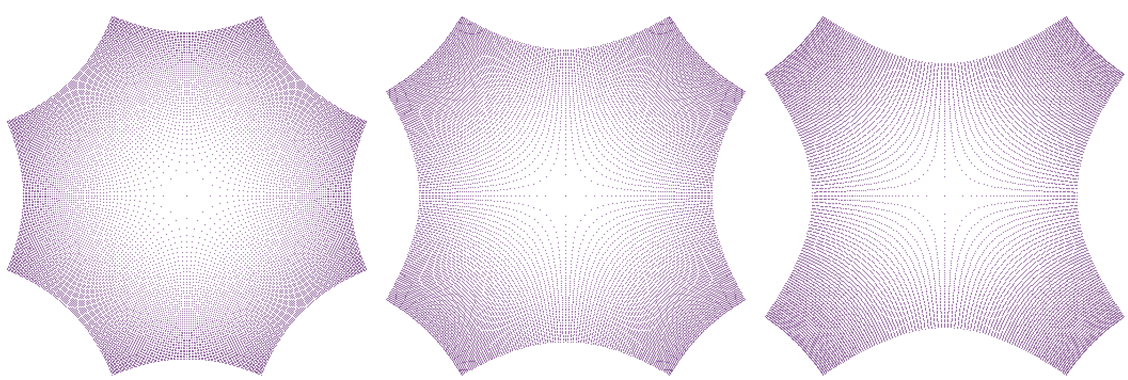

Figure 3: Stylized illustration of a quasi-conformal mapping (i.e., dilation) in complex-analytic Teichmüller theory. The analogue in a Teichmüller theory for arithmetic holomorphic structure is the theta-link.

2.2 Heights, Prime-Strips, and the Theta-Link

Prior to proving Grothendieck's anabelian conjecture (expressed by Alexander GROTHENDIECK in a 1983 letter to Gerd FALTINGS) in 1996, Mochizuki-sensei studied under Faltings as a PhD student at Princeton (and also wrote his undergraduate senior thesis with Faltings as his advisor). Interested in the work of Faltings (and Grothendieck), Mochizuki-sensei was nonetheless largely discontent with wider practices that he observed in the field of Diophantine geometry.

Mochizuki-sensei:

> Often, when people think about Diophantine geometry, they try to do things from [a] very non-canonical point of view; Vojta's-approximation-type techniques. [...] It's very computational and very non-canonical. I just didn't like that. I thought that you needed to work with canonical structures.

In the work of Faltings, on the other hand, he witnessed a mathematical style of a refreshingly canonical tenor.

Mochizuki-sensei:

> So, I was disturbed, or you maybe could even say disgusted, by the [non-canonical] sort of approach. [...] At Princeton, I ended up doing my senior thesis with Faltings. Why did I become interested in working with Faltings? At the time, Faltings was famous for his 1983 paper on the Mordell conjecture – the Mordell, Shafarevich, and Tate conjectures. [...] Faltings' approach to Diophantine geometry struck me as being much more canonical. [...] I was very much – in my mid to late teens – concerned with canonical structures.

Indeed, the theory of Faltings heights, canonical bundles, and isogenies would later find purchase in the development of IUT.

Mochizuki-sensei:

> As I've emphasized in many expositions [on] IUT, one of the main inputs to the whole structure of IUT is precisely the special case of Faltings' proof of the Tate conjecture: invariance of the height with respect to isogenies. So, this, again, is an argument that is based very much on – precisely – the canonical bundle, which is a canonical object, and working with canonical structures, as opposed to non-canonical structures.

Moreover, for Mochizuki-sensei, the mathematical contributions of Faltings and Grothendieck are of interest not solely by virtue of their respective canonicality, but furthermore, on account of the structural relationship that they enjoy with respect to one another.

Mochizuki-sensei:

> Going back to Faltings' proof of the invariance of heights under isogeny – the main idea there is to use canonical bundles; so, differentials. As I often emphasize, this corresponds precisely to étale fundamental groups and their anabelian nature.

Indeed, the theory of Faltings heights and isogenies of elliptic curves is the pedigree from which Mochizuki-sensei first begins to consider an approach to certain conjectured Diophantine inequalities, which, being well-known instantiations of greater, underlying, arithmetic structure, have served as useful motivating problems.

Those familiar with expository material on IUT – such as "Classical Roots of Inter-universal Teichmüller Theory", delivered at a UC Berkeley colloquium (via Zoom) in 2020 – will recall the key limitation that one faces when attempting to implement this approach directly for curves with stable reduction at all finite places of a number field. Unlike the case of a Tate curve, whose module of l-torsion points over a complete field admits a multiplicative subspace and canonical generator that make it amenable to the desired theta function treatment, the curves of interest over (global) number fields do not. Thus, one seeks to procure a situation in which comparable conditions hold.

Mochizuki-sensei:

> Once you do have this global multiplicative subspace and global canonical generator, you can construct a finite, truncated version of the series expansion of the theta function.

Nevertheless, there are certain accommodating situations in which a multiplicative subspace is canonically available, namely at certain primes of bad reduction. So, one wishes, somehow, to work within this accommodating situation. However, doing so is only consequential if the serviceable properties of this particularly accommodating situation can be exploited in lieu of the general situation, *pars pro toto* (i.e., using these primes of bad reduction instead of all primes).

The answer to this challenge, in IUT, is related to the structural properties of ring/scheme theories; that is to say, it involves Hodge theaters. Our introduction of the log-theta-lattice mentioned gluings between Hodge theaters; these are 'external', or inter-theater gluings. A crucial aspect of the construction of individual Hodge theaters is an 'internal' (intra-theater) gluing along what are called prime-strips. We can understand the Diophantine significance of the log-theta-lattice *vis-à-vis* the role played by prime-strips in the theory of Hodge theaters. Namely, the use of prime-strips makes it possible to simulate a global multiplicative subspace:

Mochizuki-sensei:

> Prime-strips literally [...] pick out one portion of the primes that satisfy the property that you want.

Ergo, by architecting Hodge theaters from prime-strips, one can subject the arithmetic holomorphic structure of ring/scheme theories to a Teichmüller-theoretic treatment in a manner that respects, via GMS simulation, the *pars pro toto* property that one desires between certain primes of bad reduction and the rest.

Mochizuki-sensei:

> The idea was that one needs a global multiplicative subspace. [...] This is what led to Hodge theaters. The idea is that you can have a global subspace which is the canonical one at certain bad primes, but not at most of the bad primes. In terms of distribution, you only get it right... the fraction is $\frac{1}{l+1}$ [for primes $l \geq 5$]: you only get the right answer once out of $l+1$ times. You want to summarily delete the other primes: that's the idea of Hodge theaters. So then, you can do that locally just with sets of primes, and at each local prime, you can stack up all sorts of local data. But then, you still want to have some notion of globality, because that's what the whole notion of a height is.

Meanwhile, one aspires to construct a certain isogeny, theta-function-theoretically. Taking two q-parameterized elliptic curves, one wishes to show that there exists an isogeny between them where, for each prime of bad multiplicative reduction, the q-parameter of one is equal to the q-parameter of the other raised to some large positive power. If such is the case, then the logarithmic height of one elliptic curve, times a large positive number, is roughly equal to the logarithmic height of the isogenous curve, thereby proving the height to be bounded. However, the theta function that we wish to employ for such purposes is non-scheme-theoretic, violating the structure of the ring acting on the l-torsion points of the elliptic curve: namely, it respects the multiplicative structure but violates the additive structure. Thus, one seeks to simulate a global multiplicative subspace via the theory of Hodge theaters, each with its own model of ring/scheme theory, prime-strip data, and collection of Grothendieck universes. One does so first by taking one Hodge theater in the domain of the non-scheme-theoretic gluing (i.e., the theta-link) and another Hodge theater in the codomain.

Mochizuki-sensei:

> If you are no longer working in the framework of scheme theory, you still want to do something like Hodge-Arakelov theory – but what do you do? So, Hodge-Arakelov theory is about evaluating theta functions and their derivatives at torsion points. Really, you can think about it in terms of the Fourier expansion of the theta function. It's a finite version of the usual series expansion [...] for l-torsion points. To do this expansion, you need not only the global multiplicative structure, but also, the canonical generator. This is also a part of the structure of the Hodge theater.

(Elliptic curves feature in the initial theta-data from which constructions proceed in IUT. As for the relationship between elliptic curves and anabelian geometry – one can recall that removal of the origin of an elliptic curve yields a hyperbolic curve.)

With Hodge theaters, we make our first encounter with the notion of inter-universality. As stated previously, as a matter of foundations, IUT is based on ZFCG (i.e., Zermelo-Fraenkel set theory + the Axiom of Choice + the existence of Grothendieck universes). Thus, it doesn't require or supply new foundations; it rests on foundations that will be familiar to those acquainted with scheme theory and étale fundamental groups.

Mochizuki-sensei:

> It's formulated in the same framework as SGA [Séminaire de Géométrie Algébrique du Bois Marie].

Each Hodge theater is, in turn, constructed by gluing together two components – one that book-keeps multiplicative symmetries, and another that book-keeps additive symmetries – along prime-strips. Thus, Hodge theaters are architected to allow for the deformation of ring structure in a manner amenable to the Diophantine treatment described above; they systematize both prime data (towards Diophantine ends) and multiplicative and additive structure (for Teichmüller-theoretic purposes). Thus, if one wishes to understand how a new Teichmüller theory could be serviceable to Diophantine geometry, one might begin with Hodge theaters and prime-strips (i.e., IUT I).

Distinct Hodge theaters present one with mutually alien ring theories, such that one can, in a canonical fashion, ascertain the degree to which the additive structure of one Hodge theater in the domain of the theta-link can be reconstructed in the codomain, the latter being a model of ring theory in which 'one doesn't know' the ring structure of the domain. As will be discussed in some depth in subsequent sections, reconstructive approaches to ascertaining the relationship between additive and multiplicative structure originate in anabelian geometry, i.e., one can perform certain reconstructions from étale fundamental groups. (Furthermore, the notion of mutual alienness can also be seen to be anabelian-inspired.) In this case, Mochizuki-sensei saw developments in p-adic anabelian geometry as allowing for such a reconstructivist approach that only involves the use of local objects:

Mochizuki-sensei:

> You still want to have some sort of ring structure, and you still want to have some notion of globality. So, how can you do this? It was known that if you just work with the absolute Galois group of a mixed-characteristic local field, you cannot reconstruct the ring structure. [...] So, the analogue of Neukirch-Uchida fails. [...] You want something less than the global number field, because you want to dismantle it. But you still, somehow, want the ring structure, and you want some notion of globality, in terms of heights. So, this led, first of all, to the idea that one needs to work with p-adic anabelian geometry. The absolute Galois group of a mixed-characteristic local field is not sufficiently rigid to allow you to reconstruct the ring structure, but I had this idea that if you also include the curve portion of the étale fundamental group, then you should be able to reconstruct the ring structure. [...] So, even though you have this section at the level of sets of primes, by working with these local arithmetic fundamental groups of hyperbolic curves over local fields, you can reconstruct the ring structure locally, without talking about a global ring structure. This was one important step. This happened between 2002 and 2005 or 2006.

As for the global picture, we recall the central role played in Hodge-Arakelov theory by arithmetic degrees of arithmetic line bundles, such as sheaves of square differentials. Arithmetic line bundles are of fundamental importance in that they furnish a means to express the globality of the Diophantine picture in a purely multiplicative manner:

Mochizuki-sensei:

> You can work with this multiplicative notion of line bundles, and there you get a multiplicative notion of globality – the global degree of an arithmetic line bundle. So, this is in some sense the sort of object one feels one wants to construct from the point of view of what went wrong with scheme-theoretic Hodge-Arakelov theory.

For isogenous elliptic curves, one would like their arithmetic line bundles to be approximately the same – that is, sharing a common container – despite the fact that one wants the q-parameter of one to be a large positive power of the q-parameter of the other. Of course, the arithmetic degrees of the arithmetic line bundles will not be precisely equal – some indeterminacy will remain – but one seeks to demonstrate that, even with the theta-link, one can still obtain a canonical bound. This involves the use, alongside the theta-link, of the log-link. Furthermore, one seeks to construct what is known as a multiradial representation for the canonical determination of height bounds (discussed further in a subsequent section).

Within each Hodge theater, the scheme-theoretic notions of (arithmetic) line bundles and divisors are expressed category-theoretically via the theory of Frobenioids, which also provides a structural way to distinguish étale-like (i.e., group-theoretic) from Frobenius-like (i.e., monoid-theoretic) objects. Hodge theaters are, in turn, diagrams involving various types of Frobenioids. The use of prime-strips ensures that the arithmetic line bundles under consideration reflect the *pars pro toto* prime structure of interest for simulating the GMS.

We have yet to divulge how inter-universality itself enters the picture. By way of a prefatory gesture, it might be best to first share that Mochizuki-sensei views the enterprise of simulating a global multiplicative subspace, at its very essence, as one of developing an arithmetic analogue of certain classical results in complex analysis, such as the functional equation of the theta function. Namely: the prospect of finding an identity between the desirable primes and the rest (i.e., for simulating a canonical global multiplicative subspace and canonical generator) – the *pars pro toto* situation – can be compared with the obtainment, via the functional equation of the theta function, of an identity between neighborhoods of slow convergence and rapid convergence.

Mochizuki-sensei:

> What I wrote in one of the Hodge-Arakelov papers is that this problem is very similar to the Jacobi identity and the functional equation of the theta function. [...] In the 19th century, people wanted to actually compute values of the theta function. It's very easy to compute in a neighborhood of infinity because it converges very rapidly; but it is very difficult to compute in a neighborhood of zero, because it converges very slowly. The slow convergence in the neighborhood of zero is very much like the Gaussian poles. So, you want to say that, up to a very small discrepancy, the unpleasant situation is equivalent to the pleasant situation, [...] and that's what the functional equation of the theta function says. That's what's remarkable about it.

Notably, the functional equation ($\theta(\frac{1}{\tau}) = \tau^{-\frac{1}{2}}\theta(\tau)$) is proven by verifying invariance with respect to a rotation (i.e., from τ to $\frac{1}{\tau}$), rather than, say, with respect to multiplying t by some large positive constant, which maps a neighborhood of 0 to a neighborhood of ∞. This is indeed highly analogous to the question of invariance in the log-theta lattice; one considers invariants with respect to rotations (i.e., the log-link) rather than with respect to dilations (i.e., the theta-link). The significance of this observation will become clearer as the log- and theta-links are subject to further discussion.

As will be explicated in subsequent sections, this philosophy regarding the abstraction of classical results from analysis for Diophantine purposes is a lesser-known guiding principle across the arc of Mochizuki-sensei's mathematical career, from Phillips Exeter to RIMS.

2.3 Grothendieck Connections and Multiradiality

Before proceeding to discuss the log-link or the complex-analytic motivations for IUT, we should pause and return to – and developmentally discuss further – the purpose of the theory. Certain goals have already been highlighted (e.g., disentangling multiplicative and additive structure, bounding heights). Certain readers might nonetheless be waiting for some kind of 'overall point'. In popular discourse, it has become routine to suppose the proof of the *abc* conjecture to be the 'point' of IUT: 723 pages to prove a one-line inequality. With such a supposition held, it is often asked why a theory as architecturally sophisticated as IUT is necessary to prove a statement that can be put so simply as the Diophantine inequality of the *abc* conjecture. In Mochizuki-sensei's view, however, the knowledge sought is not that formalized by any particular Diophantine inequality; rather, one wishes to ascertain the deep structural relationship between addition and multiplication underlying the *abc* and related Diophantine inequalities.

In the lexicon of IUT, one wishes to obtain what is known as the multiradial representation. Multiradiality is a conceptual centerpiece in IUT II, and so too is the multiradial representation in IUT III. When we speak of the prospect of reconstructing arithmetic holomorphic structure from the 'point of view' of a Hodge theater that 'doesn't know' the ring structure of another Hodge theater, we are giving an informal description of the multiradial representation. Here, the term multiradial is used to describe a property of arithmetic holomorphic structure that can be simultaneously accessed by all Hodge theaters from certain mono-analytic data (where, indeed, the term "mono-analytic" carries Teichmüller-theoretic connotations). With that being said, as discussed previously, reconstructability is only partial; indeed, the ultimate computation of the log-volume of the holomorphic hull amounts to an upper bound on the log-volumes of the possible images of the theta-pilot in the multiradial representation, which are subject to indeterminacies. In the context of IUT III, multiradial algorithms may be understood as yielding a canonical splitting of the log-theta lattice and constitute the key step that renders possible the log-volume computation that has Diophantine consequences. Here, canonical splittings of various types of objects – such as the decoupling of radial from coric data and the decoupling of local unit and value groups – are of eminent importance, together with the étale-picture, in the construction of certain portions of the arithmetic holomorphic structure that can be shared between Hodge theaters. At a more concrete level, certain rigidity properties (e.g., cyclotomic rigidity, constant multiple rigidity) of mono-theta environments play a fundamental role in the construction of the multiradial representation of the theta-pilot.

One can trace the genealogy of multiradiality across several chapters in Mochizuki-sensei's work. One can consider, for instance, relationships to Hodge-Arakelov. One precedent to multiradiality is that of the rigidities associated with the étale theta function. Work on the étale theta function (and tempered Frobenioids) was pursued, just prior to the writing of the IUT papers, as part of the general transition from a scheme-theoretic to a Frobenioid-theoretic approach. The étale theta function is introduced in the context of the goal of overcoming the limitations of scheme-theoretic Hodge-Arakelov theory and yields an anabelian/category-theoretic way of expressing the function theory surrounding the theta function of a Tate curve; namely, the étale theta function is amenable to a reconstructivist approach via Kummer-theoretic and anabelian techniques. Another important aspect of the theory is conjugate synchronization, which plays an important role in the context of Hodge-theoretic evaluation, and may be understood as the synchronization of basepoints of objects with distinct labels (where we observe that, *a priori*, such objects with distinct labels give rise to distinct, unrelated unrelated basepoints). Ultimately, in IUT, one applies multiradial algorithmic descent to construct a situation in which the labels of distinct Hodge theater may be omitted, namely, by quantifying the cost of doing so via the multiradial representation, which is invariant with respect to the theta-link, and related log-volume computations.

However, it might be best to consider even earlier genealogical precedents; we can go back, for instance, to p-adic Teichmüller theory. Indeed, multiradiality, as a concept, is very much related to Grothendieck's work on crystals and connections, whose influence can be seen throughout p-adic Teichmüller theory, Hodge-Arakelov theory, and IUT. Attending to this genealogy, at this point in our review, will be helpful for understanding – to return to a previous topic – a theory-building thread leading us from the prospect of an arithmetic Kodaira-Spencer theory to an inter-universal approach.

In p-adic Teichmüller theory, we encounter scheme-theoretic canonical p-adic indigenous bundles; and in Hodge-Arakelov theory, we encounter scheme-theoretic crystalline theta objects. Scheme-theoretic crystalline theta objects have vanishing p-curvature (to arbitrarily high degree), which is incompatible with the desired property of Frobenius-invariance (a property that is satisfied, for instance, by Frobenius-invariant indigenous bundles). (Indeed, the reductions modulo p of the canonical p-adic indigenous bundles in p-adic Teichmüller theory may be characterized by the square nilpotent nature of their p-curvature. Here, it is useful to recall the classical relationship between the p-curvature and the Frobenius conjugate of the Hodge filtration of a Frobenius-invariant p-adic indigenous bundle.) However, the theory of log-links and log-shells allows one to overcome the vanishing p-curvature impediment by means of mono-anabelian geometry. Étale-like log-shells play an analogous role to the section of an indigenous bundle determined by the p-curvature (i.e., the Frobenius conjugate of the Hodge filtration).

Indigenous bundles have connections which admit a section (the Hodge section) for which the Kodaira-Spencer morphism is an isomorphism; as a consequence, the Hodge filtration is not preserved by Frobenius actions. However, in the case of crystalline theta objects, vanishing p-curvature (to arbitrarily high degree) can be thought of as corresponding to some sort of counterfactual isomorphism between the crystalline theta object and the result of pulling back the crystalline theta object by an "infinite number of iterates" of the Frobenius morphism (unlike the indigenous bundle case, where one considers a finite number of iterates of the Frobenius morphism). Put another way, one does not, in a sense, expect this vanishing of the p-curvature (to arbitrarily high degree) in the case of some sort of highly canonical (hence, hopefully, Frobenius-invariant) object such as the crystalline theta object, given that the Kodaira-Spencer morphism is an isomorphism. In particular, étale-like log-shells may be thought of as a sort of resolution of this "contradiction". In the context of comparisons with IUT, the "Hodge" in "Hodge theater" is of service. (Indeed, the term is an allusion to the Hodge filtration and variation of Hodge structure.) That is to say, although the ring structures in the domain and codomain Hodge theaters of the log-link (or in the domain and codomain Hodge theaters of the theta-link) are fundamentally incompatible – a phenomenon that may be understood as corresponding to the fact that the Kodaira-Spencer isomorphism of the crystalline theta object is an isomorphism – with one another, due to the entanglement of multiplicative and additive structure, étale-like log-shells allow one to quantify and thereby encode, rather than to be hindered by, this incompatibility

During conversations with Mochizuki-sensei, it became apparent that a key impetus for the concept of multiradiality arose when contemplating the arithmetic Kodaira-Spencer morphism from the more classical viewpoint of Grothendieck connections and p-curvature. Because scheme-theoretic Hodge-Arakelov theory had failed to give rise to a suitably serviceable theory involving the arithmetic Kodaira-Spencer morphism, at some point, Mochizuki-sensei began to consider the concept of employing distinct ring/scheme-theoretic structures. Such a non-scheme-theoretic approach to developing an arithmetic Kodaira-Spencer theory seemed serviceable, as expressed in a fashion motivated by the philosophy of Grothendieck connections, for comparing mutually alien data structures. First, let us review the philosophy of Grothendieck connections. As Mochizuki-sensei described:

Mochizuki-sensei:

> [The] Grothendieck point of view is very important because it gives meaning to the notion of a connection; a connection is an isomorphism between two pullbacks. [...] So, it's literally a "connection" [...] between the fibers of two very close points.

Here, we think of our pairs of points *vis-à-vis* a fiber product of two copies of a scheme X over some scheme S (i.e., $X \times_S X$), which can be thought of as a moduli space of ordered pairs. Next, one asks how close the ordered pairs are.

Mochizuki-sensei:

> How close are they? They are close up to some ϵ, which, when

> raised to the n-th power, is zero. The most basic case is where the difference, some ϵ, when squared, is zero.

Thus, given two schemes, we learn the following from the Grothendieck connection.

Mochizuki-sensei:

> Here, we recall that it is a tautology that, *a priori*, the only thing that two distinct copies of a scheme X over S have in common is their structure as S-schemes.

Thus, for Mochizuki-sensei, the very notion of differentiation – when formulated in a sufficiently canonical fashion – bequeaths the capacity to pass between mutually alien data structures.

Mochizuki-sensei:

> A crystal essentially corresponds to a connection. So, what is a connection? A connection allows you to differentiate. [...] That's what differentiation is. It's not just some d that satisfies the Leibniz rule – it's about relating one mathematical setup to another mathematical setup.

Here, we recall that the Kodaira-Spencer morphism may be described as the derivative of the classifying morphism associated to a generically proper semi-abelian scheme of relative dimension one over some smooth curve from the curve to the moduli stack of (log) elliptic curves over the integers. It also lends itself to an interpretation *à la* Grothendieck's philosophy of connections.

Mochizuki-sensei:

> The Kodaira-Spencer map is precisely the discrepancy, relative to the connection, between the X-structure that comes from one side and the X-structure that comes from the other side. In the case of multiradiality in IUT, the X-structure here is replaced by the arithmetic holomorphic structure, or ring structure. Thus, you have the X-structure that comes from one side; the connection then allows you to compare it with the X-structure from the other side.

In p-adic Teichmüller theory, such themes manifest themselves in the theory of what Gunning called indigenous bundles. We begin with the first de Rham cohomology module of a generically proper semi-abelian scheme of relative dimension one over some smooth curve and its Gauss-Manin connection. Under projectivization, we obtain an indigenous bundle, which defines a crystal. p-adic Teichmüller theory concerns, in part, the integral (in the sense of Arakelov theory) and Frobenius-invariant nature of such bundles.

Mochizuki-sensei:

> This is very much related to the notion of differentiation. Especially in the context of p-adic Teichmüller theory, this crystal is the crystal arising from the first de Rham cohomology module of a family of elliptic curves.

To summarize: a connection, in the sense of Grothendieck, gives a canonical approach to realizing transport between two mutually alien, but infinitesimally close schemes. With this in mind, one might wonder if, perhaps, one could develop an inter-universal setup of distinct models of ring/scheme theory that are related to one another via some kind of crystal. Years later, analogies are indeed drawn in "Inter-universal Teichmüller Theory III: Canonical Splittings of the Log-Theta-Lattice" between étale-transport of multiradial containers and the theory of connections. Moreover, Mochizuki-sensei drew a genealogical thread back to Grothendieck connections during the interview.

Mochizuki-sensei:

> Ultimately, the notion of a crystal that appears in IUT is multiradiality.

The multiradial representation in IUT, at present, has a rather understated status in international mathematical discourse on the IUT texts relative to the central role that it plays in the theory. From conversations with Mochizuki-sensei, one quickly infers that the Szpiro and *abc* inequalities are regarded as mere byproducts of the acquisition of the multiradial representation in IUT – just as, for instance, the Poincaré conjecture for $D = 3$ is a mere byproduct of Thurston geometrization. Thus, notwithstanding the general media focus on *abc*, in Mochizuki-sensei's view, the multiradial representation, although never the centerpiece of popular commentary on IUT, is the greater output. In particular, it is of broader consequence than Diophantine inequalities precisely because it subsumes them within the greater enterprise of developing a Teichmüller theory for arithmetic holomorphic structure, carrying vestiges from the classical (complex-analytic) case.

Mochizuki-sensei:

> What do you learn from IUT that's more fundamental than the numerical inequality? You learn about the multiradial representation. One thing that's interesting about this point of view is that it applies in a very parallel way [...] to other Teichmüller theories. You don't really see anything that corresponds to the *abc* inequality in other Teichmüller theories. What you do see is something that corresponds to the multiradial representation; a simultaneous representation of distinct holomorphic structures – not in a completely holomorphic way, but in a way that's somehow relatively close to being holomorphic. So, for instance, in the classical complex case, you might recall that I talked about real-analytic structures, and that the ring of real-analytic functions is a *domain*. That is very closely related to the versions of Teichmüller theory that I'm currently working on. The multiradial representation is the right point of view – not just because, when you limit yourself to IUT, it describes the interesting structural result you learn – but because it translates very smoothly to other Teichmüller theories.

It is often asked why, despite the very simple nature of the statement of the *abc* conjecture, it is necessary to develop a theory as grand as that of IUT in order to prove it. We might refer to the above quote as a kind of response: the goal is to generalize the properties underlying 'what it means to be a Teichmüller theory' from complex-analytic Teichmüller theory to new settings, thus resulting in new Teichmüller theories:

Mochizuki-sensei:

> In this context, it's a very natural mode of thought to speak of "a" Teichmüller theory [...] but I think many people think of Teichmüller theory as "the" Teichmüller theory or complex Teichmüller theory. [IUT] really constitutes a very concrete example of liberating complex Teichmüller theory from analysis.

Having given such a generalization with IUT, one might then proceed to conduct an orchestra of Teichmüller theories in order to confront further conjectures, such as Grothendieck's Section Conjecture.

Mochizuki-sensei:

> On the Section Conjecture – current work with Hoshi shows

> that you can reduce the local-to-global Section Conjecture to three properties, and those three properties correspond to three versions of IUT. What the multiradial representation really tells you is that a certain aspect of the conjecture holds, and the *abc* inequality is really just a special case of that aspect of the Section Conjecture.

Although the papers on these topics are still being written, it can be said anticipatorily that in the coming years, new versions of IUT, and new Teichmüller theories, can be expected to enter the scene.

Mochizuki-sensei:

> So, with regard to IUT – I've been thinking about new, enhanced versions of IUT[,] [which] I referred to in my Gateway talk. So, that's one thing I've been working on in recent years. Another thing I've been working on that I'm giving priority to are the [...] Interface Papers. "Interface" refers to an interface between IUT [...] on the one hand and *p*-adic anabelian geometry and combinatorial anabelian geometry [on the other].

These Interface Papers are expected to be substantially shorter than the IUT papers (although, like the IUT papers, they build upon and refer back to previous work in anabelian geometry).

Mochizuki-sensei:

> One important difference between these papers and [the] IUT [papers] is: they're much shorter. There are many cases here. Each one is a slightly different theory [...] that gives rise to a different result. [...] Of course, like IUT, [each] relies on previous anabelian results.

As further theories are developed, one can reflect on IUT and *p*-adic Teichmüller theory as both belonging to a longer – again, multidecadal and pan-theoretic – endeavor to uproot complex-analytic Teichmüller theory from its analytic roots and generalize it for new purposes. Developments motivated by the philosophy underlying the theory of crystals constitute a central, and perhaps underrecognized, aspect of these new Teichmüller theories. The multiradial representation will continue to find analogues in these new theories. Thus, IUT is both a new Teichmüller theory and an indicator of a greater movement to articulate the full mathematical reach of Teichmüller-theoreticity.

Mochizuki-sensei:

> It's just one Teichmüller theory among many.

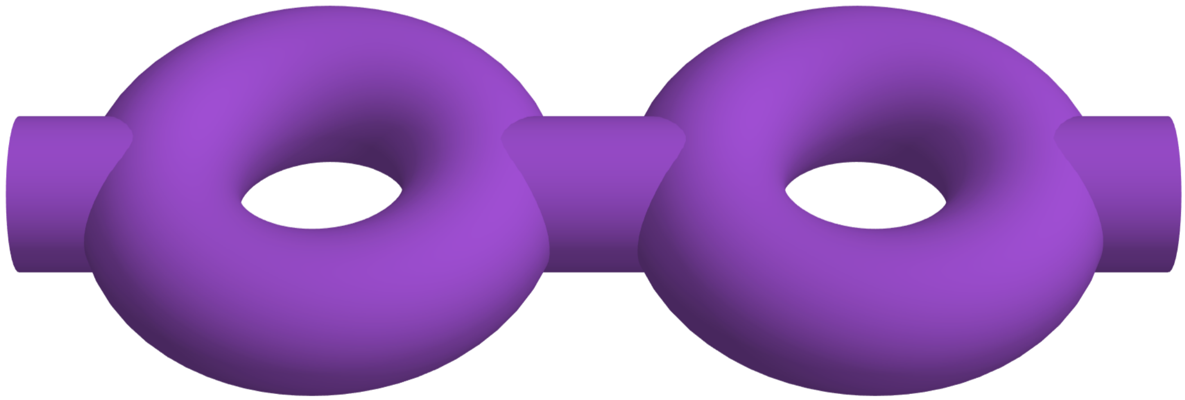

Figure 4: Stylized rendering of Figure I2.1 from IUT I, where the 2D, genus-1 surfaces are arithmetic holomorphic structures and the tubular neighborhood between them is mono-analytic.

2.4 Anabelian Origins of the Log-Link

Let's return to the log-theta-lattice. With the theta-link in play, we have multiplicative-structure-preserving and non-ring/scheme-theoretic mappings: arithmetic Teichmüller dilations. From here, the following question arises.

Mochizuki-sensei:

> The theta-link only involves multiplicative structure; so, how do you access the additive structure?

Reconstructing additive structure is of indispensable importance in examining the effect of non-ring/scheme-theoretic mappings on ring structure, that is, the effect of the theta-link on the ring structure associated to the Hodge theater in its codomain. Because the theta-link does not respect additive structure, recovery of additive structure – or, determining the bound on the extent to which the additive structure cannot enjoy recovery – permits one to calculate the distortion incurred by the theta-link. That is to say, through reconstruction, one wishes to quantify the irrecoverability of ring structure following arithmetic Teichmüller deformations: in practice, one does so by first converting additive and multiplicative symmetries into the monoid-theoretic data of groups of units and value groups.

Mochizuki-sensei:

> We have these multiplicative structures that are shared. The height portion involves value groups, but you also want to share the local units and local absolute Galois groups. So, the question is: how do you somehow access the additive structure? So, I guess, before I talk about accessing the additive structure, I should [ask] why one wants to access the additive structure. So, with multiplicative structure, there's too much symmetry. The multiplicative monoid $p^{\mathbb{N}}$ generated by p, where $\mathbb{N}$ denotes the natural numbers, is isomorphic to the multiplicative monoid $p^{N\times\mathbb{N}}$ generated by $p^{\mathbb{N}}$. There's too much symmetry, in some sense, precisely because you don't have the additive structure – you don't have the ring structure. Talking about reconstructing the additive struc-

> ture, in a situation where you already have multiplicative structure, is precisely about quantifying the situation. So, it's essentially equivalent to reconstructing a common ring structure up to some mild indeterminacy. A common additive structure is essentially the same as a common ring structure, and both are essentially the same as a quantification of the deformation.

From a combinatorial view, the local units correspond to a monoid equipped with the action of a certain profinite group (which arises from a local Galois group); the local value groups also correspond to monoids, whereas the theory of degrees of global arithmetic line bundles is expressed by means of global realified Frobenioids. The local units are preserved, up to certain indeterminacies, by the theta-link, whereas the degrees of (the Frobenioid-theoretic versions of) arithmetic line bundle are not: they are 'dilated', in Teichmüller-theoretic terms.

It had been my assumption, prior to my conversations with Mochizuki-sensei, that preoccupation with multiplicative and additive structure followed subsequently from observed shortcomings of scheme-theoretic Hodge-Arakelov theory. Discussions with Mochizuki-sensei, however, revealed the task of reconstructing additive structure to be, in fact, quite commonplace in anabelian geometry, and hence methodologically predating IUT.

Mochizuki-sensei:

> In some sense, this is a fundamental theme throughout anabelian geometry. *A priori*, you only get a multiplicative structure. So, for instance, a typical situation is that you have Kummer classes of functions or elements of a ring in some cohomology group. That gives you some sort of multiplicative monoid, and you want to construct the additive structure. This is really fundamental to – maybe not all, but most of – anabelian geometry.

A precursor to the log-link can be found in certain logarithmic aspects of the theory developed in the mid 1990's within what is known as p-adic absolute anabelian geometry.

Mochizuki-sensei:

> The development of the theory [of the log-link] was very much related to the development of absolute p-adic anabelian geometry.

Such logarithmic aspects were first discussed in an 8-page paper, "A Version of the Grothendieck Conjecture for p-Adic Local Fields". The observation underlying these logarithmic aspects is as follows. Because the p-adic logarithm converts multiplication into addition, one can apply the theory of the p-adic logarithm to reconstruct the additive structure of a p-adic field from the multiplicative structure of its units:

Mochizuki-sensei:

> In some sense, this goes back to my work in the 1990's on a version of a Neukirch-Uchida-type theorem for the absolute Galois group of a mixed characteristic local field. If you assume that the upper ramification filtration is preserved [...] you can reconstruct, from class field theory, the multiplicative units of the local field. So then, you can think of that, if you mod out by torsion, as the image [of] the logarithm – even if you don't have the logarithm, as the logarithm is a power series that requires both the additive and multiplicative structures. You can just think of that multiplicative group, modulo

> torsion, as the image of the logarithm; think of that as your additive structure, and try to construct a multiplicative structure on top of that. That was my work in the mid to late 1990's. In some sense, the [Topics in Absolute Anabelian Geometry (I-III)] papers, to a substantial extent, are an expanded version of that.

This 'multiplicative-units-mod-torsion-as-a-logarithmic-image' perspective is implemented directly in IUT for the log-link reconstruction of alien ring structure.

Mochizuki-sensei:

> So, how do you access the additive structure? The only answer that one can think of, in some sense, is the approach of this short paper from the 1990's: you just think of these multiplicative units modulo torsion as being the image of the logarithm; you think of them as additive. So, from the point of view of the log-shell, you think of the multiplicative structure at vertical coordinate 0 as the additive structure at the vertical coordinate 1, and you have to deal with the issue of how to reconstruct the multiplicative structure at the vertical coordinate 1. I guess that explains, essentially, why one wants to think about log-links: it's all about quantifying the deformation.

The log-shell, in essence, serves as a container for the possible images of the p-adic logarithmic applied to one's local units. Tensor packets of log-shells appear in the multiradial representation of the theta-pilot and are subject to various indeterminacies. Since the logarithm converts additive to multiplicative structure, such indeterminacies reflect and indeed quantify the 'entanglement' between these structures. Thus, the indeterminacies in IUT, far from being a nuisance or hindrance, may be interpreted as expressing Diophantine constraints; the 'noise', as it were, is informative.

The construction in IUT of a common additive structure, up to relatively mild indeterminacies, for the ring structures associated to the domain and codomain Hodge theaters of the theta-link is a highly nontrivial procedure; one must construct invariants with respect to the log-link. Such invariants include log-shells, log-volumes, and – with a nod to anabelian geometry – étale-like structure. Within the log-theta-lattice, for a given theta-link, one must, as it were, extract certain invariants from entire vertical columns on either side – the log-invariants – that are amenable to transport. The situation of interest is known as the "infinite H" – with two infinite log-link columns on either side of a single theta-link.

Mochizuki-sensei:

> Precisely because the theta-link is not compatible with the additive structure, you have log-links on both sides, and they're not compatible: this is the "infinite H". The vertical line on the left has nothing to do with the log-links on the right. And, so then, this leads one to the question of how to form invariants with [respect to] the logarithm. So, in the infinite H, if you can relate things on the left- and right-hand side of the theta-link [...] that only involve the multiplicative structure. [...] What we really want to think about [are] structures at other vertical coordinates. How do we squeeze them into this narrow tunnel so we can send them over to the other side? This is precisely the issue of thinking about invariants with respect to the log-link.

To briefly return to a previous topic, we can now offer a simpler remark on the multiradial representation: it is constructed by means of invariants with respect to the log-link, i.e., data arising from a vertical column of the log-theta-lattice that is independent of the vertical coordinate.

The infinite H "tunnel" constituted by the theta-link is traversed by using three types of data that are invariant with respect to the log-link:

Mochizuki-sensei:

> We're interested in structures that are invariant with respect to the log-link. The first one is the étale-like structure: anabelian objects that are reconstructed. Then, upper semi-commutativity arising from the log-shell: that's the next one. Invariance of the log-volume is another important [one]. So, these are the three invariants: holomorphic étale-like structures; upper semi-commutativity with respect to the log-shell; and the invariance of the log-volume. So, one has these three structures that are invariant with respect to the log-link. One wants to use these to reconstruct, or to construct, some version of the ring-theoretic data that [is] closely related to the data [in another Hodge theater] – but there's some distortion.

The origins of upper semi-commutativity may be traced back to the following considerations in p-adic anabelian geometry:

Mochizuki-sensei:

> The most fundamental invariant is the étale-like structure, the local arithmetic fundamental groups, which also play a key role in the structure of Hodge theaters. This is closely related to absolute p-adic anabelian geometry – this is why it appears in Absolute Topics III. [...] So, the absolute mono-anabelian p-adic geometry in Absolute Topics III gives you a container, a common container, for one vertical line of the log-theta-lattice. So then, you have Kummer maps at each vertical coordinate, but there's this distortion arising from the log-link. So you want some sort of relationship between the Frobenius-like data at each vertical coordinate and the invariant data, the étale-like data. So, the tautological solution is the log-shell. If you apply the Kummer map one way, and take the log and apply the Kummer map again, you get [...] literally the p-adic logarithm. There's no obvious relationship between going one way and going the other way.

Due to the non-commutativity of the log-theta-lattice, one wishes not to quantify distortion by arbitrarily taking particular links or `paths' within the lattice; rather, one seeks a canonical upper bound that captures the greatest distortion one can encounter.

Mochizuki-sensei:

> You just take the point of view that you are going to think of things in a very rough way – no matter which way you go, you have this upper bound: it's just a log-shell.

As mentioned, the relationship between absolute p-adic anabelian geometry and the log-link is first discussed in detail in "Topics in Absolute Anabelian Geometry III", a paper which can be thought of, perhaps, as a kind of prequel to the four IUT papers.

Mochizuki-sensei:

> Absolute p-adic anabelian geometry later results in these Absolute Topics I, II, and III papers. Something like the log-link first appears in Absolute Topics III. [...] I released those papers in the spring

> of 2008. I started writing the IUT papers shortly after that, in the summer of 2008. I finished writing them essentially around 2010 or 2011 – quite a long time ago, actually...

Thus, within a modest uncertainty interval, we can estimate the period during which the log-link was conceptualized.

Mochizuki-sensei:

> It was sometime between 2002 and 2006, I guess.

The close relationship between the development of the log-link and absolute p-adic anabelian geometry raises, for those interested in the history of IUT, a further question about the theory-building pathways that Mochizuki-sensei followed in the 2000s. Indeed, alongside IUT, Mochizuki-sensei also developed absolute anabelian geometry and combinatorial anabelian geometry. From a contemporary vantage point, such theories appear to follow tandem, interrelated, but nonetheless distinct theoretical trajectories. I never would have presumed them to be, in some sense, coterminous; I had expected them to constitute a parallelized distribution of Mochizuki's interests, spread over multiple theories.

What I discovered, to my surprise, was that IUT, precursor papers to IUT (e.g., on Frobenioids), absolute anabelian geometry, and combinatorial anabelian geometry were all initially conceptualized as a singular theoretical colossus, which, for practical reasons, was subsequently anatomized into various mechanisms and theoretical architectures, published in serial installments.

Mochizuki-sensei:

> Many of the papers written during this period, such as semi-graphs of anabelioids or the étale theta function – originally, all of these papers, together with the four IUT papers – were intended to be a single paper. And I started writing, and then I realized it was going to be too long. And so, I had to split it up into more papers, and it kept splitting and splitting and splitting; and that is what resulted in all of these papers. [...] Also, the Absolute Topics papers; they were all intended to be a single paper.

Thus, when it is opined by readers that the terminology of the IUT papers appears to 'come out of nowhere', I wonder if it would be helpful to view the IUT papers as volumes belonging to some kind of greater textual series. Without reading the 'IUT prequels', such as the Topics in Absolute Anabelian Geometry (I-III) papers, one is beginning *in media res*.

However, such partitioning was not, in fact, arbitrary. The theories and concepts did, as I suspected, emanate from distinguishable motivations, and thus could be subject to independent treatment.

Mochizuki-sensei:

> This is a very interesting question. Even though they all originated as a single paper [...] many aspects of the Absolute Topics papers – and also the combinatorial papers, and so on – did have independent origins. Even though they were originally intended as a single paper, it was intended [to be] a single paper that was not just about the *abc* inequality, but about this whole circle of ideas. So, it's sort of interesting that the recent papers that I've been working on, the Interface Papers, have come back to the roots of this whole circle of ideas. So, they are in service of IUT, but IUT is also in service of them; it's gone full-circle.

I'm still not equipped to better describe, in full, the common origin of IUT, absolute anabelian geometry, and combinatorial anabelian geometry, nor the manner in which they have ramified. However, the intended purpose of the Interface Papers that Mochizuki-sensei is currently writing is to make visible the common webbing that hangs betwixt these theories.

Mochizuki-sensei:

> That's sort of the point of the Interface Papers.

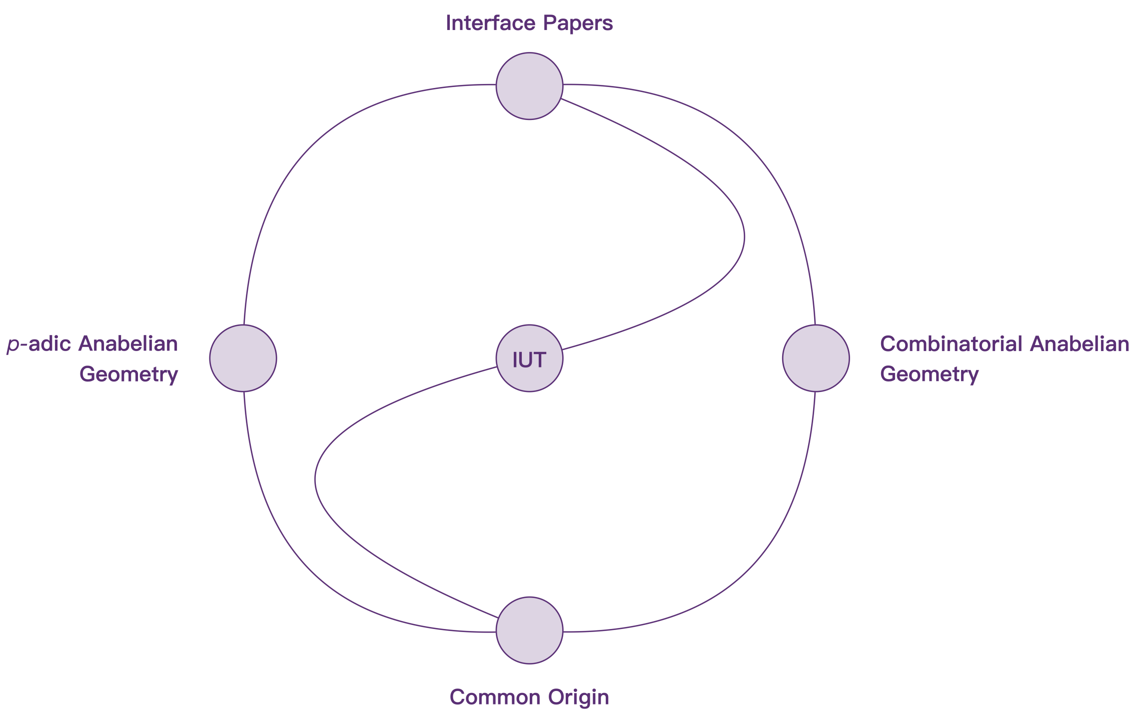

Figure 5: "Metamathematical cartoon" of the relationship between IUT, *p*-adic anabelian geometry, and combinatorial anabelian geometry, and their synthesis in the forthcoming Interface Papers.

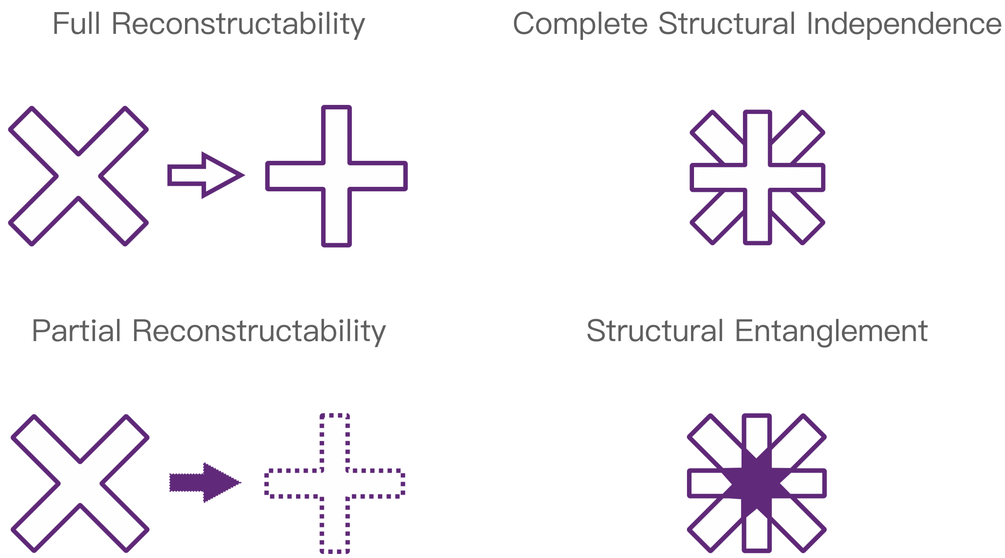

Figure 6: Relationship between anabelian reconstruction and ⊠/⊞ entanglement.

2.5 Canonical Coordinates and Coricity

It has been previously mentioned that further remarks on the relationship between IUT and prior results in classical analysis will be given. We can now – with Hodge theaters, prime-strips, the theta-link, log-link, log-theta-lattice, and multiradial representation having been discussed – commence our turn towards this topic, which will be discussed over the course of the next few subsections. As suggested previously, one underlying principle guiding the enterprise of abstracting certain results from classical analysis is the pursuit on Mochizuki-sensei's part of canonicality. Thus, by way of beginning a turn towards our discussion of analysis and canonicality, let's start with the topic of canonical coordinates, which again brings us back to anabelian geometry.

The issue of canonical coordinates constitutes another recurrent theme across the theories of Mochizuki-sensei. As one can read in "An introduction to p-adic Teichmüller theory", for instance, the foremost objective under pursuit is the uniformization of p-adic hyperbolic curves, and their moduli, thereby generalizing, to the nonarchimedean case, the complex uniformizations of Koebe/Fuchs and Bers. Recall that uniformization of Riemann surfaces endows them with canonical coordinates. In the case of p-adic Teichmüller theory, the objective is the attainment, in the p-adic case, of canonical coordinates from Frobenius liftings.

An earlier encounter with the theme of canonical coordinates can be found in a review article, "The Grothendieck Conjecture on the Fundamental Groups of Algebraic Curves", published in *Sugaku Expositions* by the AMS, which Professor NAKAMURA Hiroaki (中村博昭) and Professor TAMAGAWA Akio (玉川安騎男) coauthored with Mochizuki-sensei. In the review, Nakamura-sensei, Tamagawa-sensei, and Mochizuki-sensei discuss the relationship between Koebe uniformization and anabelian reconstruction. Namely, given a complex hyperbolic curve, which defines a Riemann surface, uniformization of the Riemann surface allows one to construct a canonical representation of its (topological) fundamental group in $SL_2(\mathbb{R})$. Likewise, when simply given no more than the representation, one can recover the fundamental group action on the upper half plane, and, from the quotient of the upper half plane by this action, recover both the Riemann surface and the original hyperbolic curve. Moreover, this Koebe/Fuchs uniformization by the upper half-plane is related to the Bers uniformization of the moduli space of curves. Given the title of the expository article, it is natural that one might view Koebe uniformization from the vista of reconstruction: anabelian geometry is, after all, reconstructive.

The theme of canonical coordinates continues to percolate, in various visages, throughout subsequent works by Mochizuki-sensei. For instance, one might seek to obtain canonical coordinates for arithmetic fundamental groups or Galois categories, within the framework of absolute anabelian geometry, independently of the choice of basepoint. Such is a guiding aspiration of the theory of anabelioids. In the case of anabelioids, fundamental groups may be canonically constructed from faithful quasi-cores. Readers of the IUT papers may recognize faithful quasi-cores as precursive to the concept of coricity, a key principle accompanying that of multiradiality in IUT.

Moreover, basepoint-independence is a paramount criterion in IUT, for the basepoints of the étale fundamental groups associated with distinct models of ring/scheme theory (corresponding to Hodge theaters) are unrelated. Basepoint-sensitivity is a key impetus for inter-universality; each basepoint corresponds to its own set-theoretic universe. As for coricity: Galois groups, *qua* abstract groups – that is, deracinated from any underlying field structure – that are coric with respect to the log-link can permeate what is known as the 'log-wall' separating mutually alien ring structures, thereby helping to facilitate reconstruction of the ring structure of another Hodge theater. This, in turn, is likened, in "Topics in Absolute Anabelian Geometry III", to the use of Frobenius liftings in p-adic Teichmüller theory for the construction of canonical coordinates. Splitting monoids at primes of bad reduction are likewise likened in IUT III to the construction of canonical coordinates in p-adic Teichmüller theory.

Thus, one may surmise a relationship between canonicality of coordinates and reconstructability – a surmisal to which Grothendieck's anabelian philosophy leaves one particularly predisposed. IUT, by invoking certain perspectives and resources that originate in precursor works such as p-adic Teichmüller theory and the theory of anabelioids, allows canonicality of coordinates to achieve an inter-universal register in its relation to the reconstruction of mutually alien ring/scheme theories.

Mochizuki-sensei:

> What's the point of thinking about canonical uniformizations or canonical labels? You want to understand the object – [this is] the whole reconstruction aspect of anabelian geometry – from the point of view of an alien intelligence: someone who doesn't know about the setup.

This genealogical yarn finds an inter-universal manifestation in the concept of coricity.

Mochizuki-sensei:

> You want to formulate structures in such a way that, somehow, you can't tell the difference between [the] two. [...] This is about canonicality, or, you could say, coricity: you want [...] to focus on the portion of the structure that is such that you can't tell the difference whether you're dealing with this or [that]. So, [this] was the birth of inter-universality.

In the context of the theta-link, coric data refers to the mono-analytic data that is shared between Hodge theaters in the domain and codomain of the theta-link, while multiradial data refers to data arising from the arithmetic holomorphic structure of a Hodge theater in the domain or codomain of the theta-link that may be reconstructed, up to mild indeterminacies, from the coric data. Here, the theory of canonical coordinates and the theory of crystals come together.

Although anabelian reconstruction has been underscored herein, as a historical matter, the topic of canonical coordinates does not actually begin, for Mochizuki-sensei, with anabelian geometry. We must, in fact, revert back to the early 1980's – Mochizuki-sensei's high school days.

2.6 The Gaussian Integral and Other Analytic Motivations

Readers of expository material on IUT written by Mochizuki-sensei will likely have encountered references to the Gaussian integral. As a reader, prior to interviewing Mochizuki-sensei, I had presumed that the Gaussian integral was a kind of pedagogical device whose expository affinity with certain themes in IUT had been ascertained *ex post facto*. What I learned instead is that it was, in fact, an early mathematical fascination of Mochizuki-sensei's, which has left a formative imprint on his theory-building arc.

Mochizuki-sensei:

> Since I was 13 or 14, I [have been] very much fond of the Gaussian integral.

In fact, it was the subject of Mochizuki-sensei's first 'talk'.

Mochizuki-sensei:

> When I was 14, I entered Phillips Exeter Academy, in Exeter, New Hampshire. [...] When [...] I entered the Phillips Exeter Academy, there were these clubs, and there was something called the math club. So, I became a member of the math club and I gave a talk on [the Gaussian integral].

Boyd:

> Was the change of coordinates of interest to you?

Mochizuki-sensei:

> Yes, the change of coordinates.

Recall that the Gaussian integral, which is refractory to integration by usual methods, can nonetheless be made amenable to integration by first taking two copies of the integral and converting to polar coordinates. This struck Mochizuki-sensei as more canonical, as it had little to do with explicit integral computations. By cloning the integral and finding the right choice of coordinates, the computation becomes trivial.

Mochizuki-sensei:

> What's amazing about this computation is that, if you square this [pointing to the integral] and use polar coordinates, this can now be integrated.

In IUT, the legacy of 'cloning' as such can be seen as inherited by the use of mutually alien ring structures (i.e., mutually alien copies) – that is, a multitude of Hodge theaters in the log-theta-lattice. The change of coordinates, from Cartesian to polar, can be analogized to the conversion from Frobenius-like (i.e., monoid-theoretic Frobenioid portions) to étale-like objects (e.g., Galois/fundamental groups). This involves, in fact, a revision of Hodge-Arakelov evaluations, recast in the framework of mono-theta environments via Kummer-theoretic relationships between Frobenius-like and étale theta functions, where Hodge-Arakelov-theoretic evaluation gives rise to theta monoids, Gaussian monoids, and theta-pilots in the domain of the theta-link. (Such is the crux, in a sense, of IUT II.)

Here, the term "Gaussian" in Gaussian monoids is chosen because it refers to the domain of the theta-link, whose similarity to the Gaussian distribution is discussed in a later subsection. These, in turn, are developed in the framework of log-shells (as logarithmic procession Gaussian monoids, or LPGs) in IUT III. Log-volumes are obtained from these LPGs via multiradial algorithms, with Diophantine consequence.

However, the Gaussian integral is not the only classical-analytic result to find arithmetic reincarnation in IUT; recall the functional equation of the theta function. Indeed, Mochizuki-sensei's fascination only intensified as he found that the intriguing quality of the Gaussian integral was not a standalone instance, but could instead be found to echo across many results in analysis:

Mochizuki-sensei:

> A more advanced version of this is the functional equation of the theta function. [...] When I was 16, at Princeton University, I took a course on Fourier analysis given by Joseph Kohn, a famous analyst. [...] It covered [....] the functional equation of the theta function. This can be sort of regarded as a function-theoretic version of [the] Gaussian integral.

Just a few years later, an interest in (complex-analytic) Teichmüller theory began.

Mochizuki-sensei:

> I entered graduate school at Princeton when I was 19, and [...] around the age of 20 or so I became familiar with complex Teichmüller theory.

With canonical structures in mind, a growing fascination with such analytic results might lead one to think about canonical coordinates.

Mochizuki-sensei:

> The issue of uniformization, reconstruction, and canonical labels – it goes back to these kinds of topics, because these kinds of topics are concerned with grasping the objects under consideration in a canonical way. This turns out to be related to IUT later because the alien observer is the person on the other side of the Fourier transform. The whole content of [the] Jacobi identity is that this Gaussian is invariant under the Fourier transform. The whole point of all of this is that you want to work with structures that are invariant – visible – to alien observers.

By an "alien observer", Mochizuki-sensei has in mind a mathematician who, although without direct access to a mathematical object, is nonetheless able to recover, reconstruct, or understand it, on the basis of very minimal input data, by merit of its canonicality, without superfluous or ancillary data. One could use "reconstructor", perhaps, as a synonym; however, such a reconstructor should be one who shares no mutual information with the mathematician who created the mathematical data, other than the minimal information (i.e., input data) needed for reconstruction. For instance, with Grothendieck's anabelian conjecture proven, one knows that another mathematician – perhaps from a future culture – can fully discover the wonders of proper, smooth, hyperbolic curves over number fields so long as they have étale fundamental groups. This theme attains an inter-universal register in IUT, where the ring structure is constructed in one Hodge theater only using the input data that can be passed from another Hodge theater.

What came to light during the discussion is that certain novel methods utilized to fashion an imbrication between Hodge-Arakelov

Theory and inter-universal geometry were sourced – or "liberated" – from complex analysis. That is to say, Mochizuki-sensei's response to the issue of Gaussian poles, which were named in a manner invocative of Gaussians, was indeed a kind of abstraction of certain results from 19th-century analysis.

Mochizuki-sensei:

> Hodge-Arakelov theory [...] looks very tempting, because it almost looks like it's going to imply the Szpiro conjecture for number fields, but the problem is these Gaussian poles. So, what I wrote in one of the Hodge-Arakelov papers is that this problem is very similar to [the] Jacobi identity and the functional equation of the theta function.

This aspect of the transition from Hodge-Arakelov theory to inter-universal geometry can be perhaps understood as repurposing and abstracting certain results from complex analysis, including the Gaussian integral, the functional equation of the theta function, and complex-analytic Teichmüller theory.

Mochizuki-sensei:

> I was fascinated by these phenomena in analysis. They seemed very opaque to me because they were formulated in the framework of analysis. One thing I always had trouble with in analysis is precisely this non-canonicality. Everything is $\mathbb{R}$ or $\mathbb{C}$, [but] it seemed to me that some $\mathbb{R}$'s are different from other $\mathbb{R}$'s, and they really need to be distinguished. The fact that everyone said "everything is $\mathbb{R}$" – that just disturbed me. So, one fundamental theme, from my point of view, was: I had this idea that these phenomena are fundamentally phenomena that have nothing to do with analysis. We're just looking at a rough shadow of the actual phenomena when we're looking at the formulation in analysis. [...] It's not the real, actual phenomena that we actually want to look at.

One might posit that *p*-adic Teichmüller theory was already indicative of such an imperative. However, this too, like Hodge-Arakelov theory, was dissatisfactory.

Mochizuki-sensei:

> *p*-adic Teichmüller theory is one approach to this, but it's still not as abstract as I wanted it to be.

Thereafter, the enterprise was set afoot to repurpose certain results from complex analysis for application to Diophantine geometry, or to –

Mochizuki-sensei:

> liberate important results in analysis from analysis.

Thus, the answer to the question that had left me vexed, as I vied to ascertain the principles underlying the transformation from Hodge-Arakelov theory to IUT during the 2000-2004 period – much to my surprise – was that of rendering abstract, for arithmetic treatment, the recurrent phenomenon that Mochizuki-sensei had witnessed in analytic results.

Mochizuki-sensei:

> The real phenomenon, in its full glory, can only be seen when formulated in an abstract fashion that has nothing to do with analysis [...] So, then I started looking for such an abstract arithmetic version [...] around 2000.

It is with this principle in mind that the motivation for inter-universality can now be explicated.

2.7 $\in$-Loops and the Membership Equation

By 2004, as shared in his Tokyo lectures, Mochizuki-sensei had forsworn a scheme-theoretic approach, concluding that, when wrestling with the impediment of Gaussian poles, one is necessarily obliged to confront the "membership equation": $a \in a$. Of course, such a relationship – an element being a member of itself – violates the set-theoretic Axiom of Foundation. Nevertheless, Mochizuki-sensei proceeds, in the 2004 lectures, to offer a preliminary, prototypical sketch of his inter-universal program for attending to the $a \in a$ issue. Readers of "Inter-universal Teichmüller Theory IV: Log-Volume Computations and Set-Theoretic Foundations", will recall that the theory of species are introduced in order to permit one to "simulate $\in$-loops without violating the axiom of foundation". Despite my best efforts, I was unable, prior to my arrival in Kyoto, to identify the origin of the membership equation or the issue of $\in$-loops. Nevertheless, Mochizuki-sensei explained its origins in detail.

The membership equation characterizes a situation in which the information or property of interest innate to a whole object can likewise be found equivalently in its part. I'm reminded of the Latin expression for such a situation – *pars pro toto*: a part for the whole. (One might recall previous use of this expression with respect to prime-strips; this is not coincidental.) For Mochizuki-sensei, "$a \in a$" is the most succinct encapsulation of a *pars pro toto* relation: if one finds that the part is equivalent to a whole, such implies that, with the part being a member of the whole, the part is also somehow a member of that same part, a member of itself: the $\in$-loop ensues. Moreover, this phenomenon can be found in numerous mathematical topics: the functional equation of the theta function, Grothendieck's Section Conjecture, Frobenius liftings, and the Taniyama-Shimura (i.e., modularity) conjecture. Furthermore, in IUT, one witnesses the manifestation of this *pars pro toto* principle in the reconstruction of arithmetic holomorphic structure itself. (Recall that the prefix "holo-" derives from the Greek "ολος" (hólos), meaning "whole".) Indeed, anabelian reconstruction of arithmetic holomorphic structure can be thought of as reconstituting the whole (i.e., the rigid coupling of two underlying combinatorial dimensions) from an underlying part (i.e., one underlying combinatorial dimension). One might recall multiradiality and mono-analyticity in this respect.

By way of discussing the greater theme of the membership qquation, let's begin with a somewhat contemporary milestone in number theory: Taniyama-Shimura. Elliptic curves are parameterized by modular curves. Yet, the Taniyama-Shimura conjecture, proven by Wiles and Taylor (thus implying Fermat's Last Theorem, thanks to Ribet), states that certain specific elliptic curves are in fact modular curves, that is, the moduli space and the point are equivalent.

Mochizuki-sensei:

> One example of this is some sort of equivalence between a curve – a specific curve – and the moduli of all curves. So, *a priori*, this [curve] should be a member of this [moduli space], but then to say that there's some sort of equivalence is to say that [$a \in a$]. This phenomenon can be seen in Wiles' work on the modularity conjecture – because there, the whole point of modularity is to get a map from a certain finite étale covering of the moduli space of elliptic curves to a specific elliptic curve.

The Gaussian integral itself is another example. Conversion from Cartesian to polar coordinates allows one to radically simplify the computation of the integral by applying a change of variable, i.e., substituting e^{-r^2} for a new coordinate u.

Mochizuki-sensei:

> This, in itself, is sort of an example of this [membership equation] because [...] in set theory, exponentiation [i.e., e^{-r^2}] corresponds to the set of subsets in a set, so this is always going to be of larger cardinality. This kind of issue is also very much related to some sort of equality or equivalence between these two. So here, if you make a change of variable [i.e., substituting e^{-r^2} for u], this is precisely, formally, this kind of situation [pointing to the membership equation]. And this ultimately corresponds in IUT to the theta-link.

Thus, one might say that when Mochizuki-sensei invokes this issue of $a \in a$, his intent is to capture, in the simplest possible statement, this pattern, one to be found reverberating intergenerationally throughout mathematics.

Mochizuki-sensei:

> All of these are examples of a part being equivalent, somehow, to the whole. And that's what this membership equation is. That's what these $\in$-loops are.

As mentioned previously, $\in$-loops are simulated by species, the subject of §3 in IUT IV. 'Species theory' is an algorithmic framework for handling data belonging to different (Grothendieck) universes within ZFCG, independently of the ZFC model chosen in a given universe. Thus, a species abstracts the notion of a type of mathematical object – a collection of formulae – that traverse universes in a model-agnostic way. Although species theory received a certain degree of attention in the early days of public discourse on IUT (*circa* 2012-2014), public discussion of species theory has become less noticeable in the intervening years, perhaps because attention is largely concentrated on IUT III.

Perhaps this abstract-algorithmic framework can be motivated by giving a species-theoretic interpretation of a well-known membership-equation-type result. Thus, I asked Mochizuki-sensei how one might interpret the Taniyama-Shimura conjecture in terms of the IUT IV framework.

Mochizuki-sensei:

> What does the statement of the modularity conjecture look like from the point of view of species? Right, so – [...] it's an equivalence between an elliptic curve and the moduli of elliptic curves, and that looks like the membership equation. This is what I thought was interesting. [...] The way you overcome the membership equation is by working with species – so, in other words: you have an abstract notion of some type of mathematical object, and it can be realized in various models of set theory. From the point of view of this situation that occurs in the statement of the modularity conjecture: you have a contradiction to the membership equation if you think in terms of an equivalence between a single elliptic curve and the moduli of elliptic curves, but I guess the "species" one is using to relate the two is the notion of a curve. The elliptic curve is a particular curve, and the modular curve is a particular curve. So, the "species" that is being used here to overcome the membership equation is the notion of the curve. You get a literal contradiction to the membership equation if you think of the elliptic curve as one elliptic curve and the modular curve as a set of elliptic curves; then you get a literal contradiction to the membership equation. This is, I guess, an interesting point – why do you not get a literal contra-

> diction to the membership equation? It's because you're thinking in terms of the type of mathematical object, which is the curve, and you're relating the underlying curve of an elliptic curve to the underlying curve of a modular curve.

Here, one can think of the algebraic varieties involved algorithmically in terms of curves, rather than particular sets of solutions. A species is, effectively, an abstraction of this principle to a general algorithmic framework: it gives one a type of mathematical object, which is not itself a set, such that membership-equation-type relations between data in different universes can be established without precipitating the ∈-loops that would arise if one worked directly with sets. However, 'species theory' is not a new foundational framework for mathematics; ZFCG provides the foundations. Rather, species-theory can be viewed as an explicit guided tour as to how, in effect, algorithmic inter-universality can exploit nested universes to evade the kind of paradox that would ensue if one worked only with sets.

Mochizuki-sensei:

> With regard to the use of species in IUT, I emphasize this in a number of places in the fourth paper: from my point of view, there's nothing radical about species; it's just an explicit way of recording operations that mathematicians do without explicitly recording them.

Put differently, 'species theory' could be analogized to a hypothetical appendix to Taniyama-Shimura in which it is assured that apparent set-theoretic paradoxes are eschewed thanks to the providence of the notion of a curve.

Expository Example	
	Modularity conjecture: elliptic curves and moduli spaces
Classical–Analytic Examples	
	Functional equation of theta function
	Gaussian integral
Teichmüller–Theoretic Examples	
	Frobenius liftings (p-adic Teichmüller)
	The theta–link (IUT)
	Reconstruction of arithmetic holomorphic structure (IUT)

Figure 7: A table of ∈-loops discussed with Mochizuki-sensei.

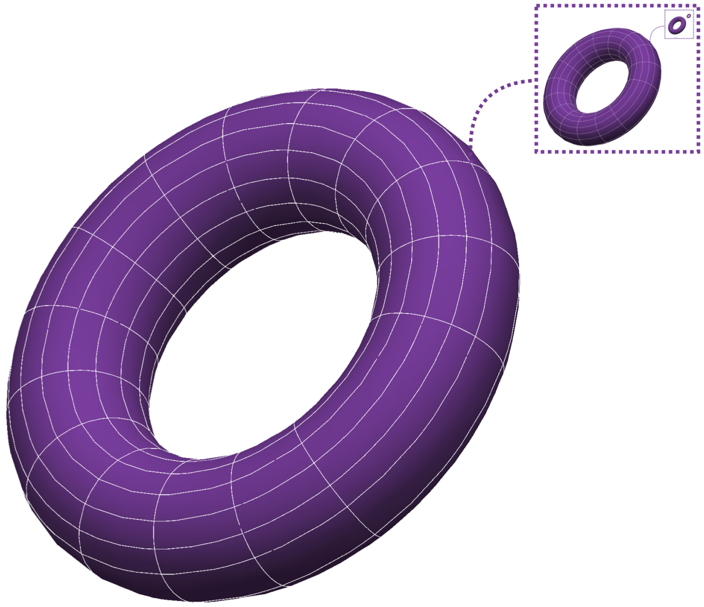

Figure 8: An illustration of the $\in$-loop at play in Taniyama-Shimura, whose set-theoretic contradiction is eschewed thanks to what, according to IUT IV, one might call the species of curves.

Reading the four IUT papers, one might take note of the presence of two simulatory concepts. One, discussed herein, is that of species (from IUT IV), which simulate $\in$-loops. The second is that of prime-strips (from IUT I), the arithmetic local-analytic sections that simulate the global multiplicative subspace. As it so happens, the respective introductions of these concepts roughly bookend the PRIMS special issue, with relatively little crosstalk; that is to say, the construction of species does not make mention of prime-strips, nor *vice-versa*. Nonetheless, one might suspect that the common simulatory nature of the theoretical purposes vested in them might somehow endow them with a certain theoretical kindredness. So, I asked Mochizuki-sensei about this.

Mochizuki-sensei:

> So, you're looking at this common simulation aspect. [...] So, here [pointing to species] one is talking about the foundational structure of the theory, and this [pointing to prime-strips] relates to the content of the theory. I didn't really think about this in this way before. So, this is actually interesting. [...] It's a new discovery.

Several weeks later, we returned to the subject. Beginning with the nature of modular curves in the Taniyama-Shimura conjecture, we came to discuss the relationship between $\in$-loops and differentials.

Mochizuki-sensei:

> From this point of view, I think it's an interesting question. Going back to what you were saying during our first meeting when you talked about species, on the one hand, as ways of simulating the membership equation, and prime-strips as objects that are used to simulate the GMS: it's not just a coincidence – they are closely related, because the way prime-strips are used to simulate the GMS

> is very much a membership equation type of situation. The use of prime-strips to simulate a GMS basically amounts to – [...] you want to choose one particular prime-strip that has the right properties: that's the idea of simulating the GMS. [. . .] Really, the notion of a section is really a membership equation type of situation.
>
> Getting back to your question about the membership point of view vs. the differential point of view [...] you can obviously see the roots of IUT in *p*-adic Teichmüller theory. So, differential geometry is very closely related to Frobenius actions. This is sort of the main point of view of *p*-adic Teichmüller theory: the hyperbolic metric on the upper half plane is analogous to Frobenius. The Weil-Petersson metric on the moduli stack of curves should be thought of as analogous to a canonical Frobenius action. So, what is a Frobenius action? So, for a hyperbolic curve, there is no lifting of Frobenius. modulo *p*, the Frobenius is precisely a membership equation. The multiradial representation is like a Frobenius lifting. They're all membership equations.

Thus, the simulatory roles of species and prime-strips are, as we uncovered, very much related. In some sense, their simulatory roles offer two views on IUT, which happen to coincide: the algorithmic view (species) and the data-structural view (prime-strips). Moreover, this discovered kindredness allows one to thematically translate the PRIMS special issue across itself – viewing IUT I from the vista of IUT IV, and *vice-versa*. Put differently, prime-strips situate the membership-equation simulation in the register of Teichmüller-theoreticity (i.e., for arithmetic holomorphic structure), whereas species situate the membership-equation simulation in the register of inter-universality, with prime-strips and species relating to one another inside each Hodge theater.

Mochizuki-sensei:

> So, it's interesting – because of the way that the culture has developed, we have different names: the membership equation; the notion of the section, or even the section conjecture; or the notion of a Frobenius lifting; or a prime-strip. They look like completely different, unrelated objects – but they're really the same phenomenon.

In many respects, the discussions I had with Mochizuki-sensei concerned not only IUT, but a kind of greater enterprise to which IUT and preceding theories belong. Such is especially the case given the original plan to package many of Mochizuki-sensei's works from the 2000s in a single paper; what was originally a massive theory is now a greater, textually-distributed intellectual enterprise.

Having reviewed the species-theoretic character of inter-universality and its relationship to the Diophantine motivations of the theory, let us conclude with the notion of IUT as a Teichmüller theory for arithmetic holomorphic structure. The enterprise of generalizing and abstracting, beyond analysis, 'what it means to be a Teichmüller theory' also requires that one generalize and abstract, beyond analysis, the very nature of holomorphy. In closing, we'll briefly return to the relationship between arithmetic holomorphic structure and class field theory, and the corresponding emergence of the log-theta-lattice from the development of a Teichmüller theory for arithmetic holomorphic structure.

Concept	Prime–Strips	Species
Description	Local analytic section	Model–agnostic collection of set–theoretic formulae
Simulation Provided	**Simulate** the GMS	**Simulate** $\in$–loops without violating axiom of foundation
Role in IUT	Arithmetic content	Algorithmic structure
Theoretical Theme	Teichmüller–theoreticity	Inter–universality
IUT Paper	IUT I	IUT IV

Figure 9: A conceptual comparison of the simulatory role, theoretical theme, and textual source of prime-strips and species.

2.8 Arithmetic Holomorphic Structure

Much of modern mathematics is an enterprise in abstract elaborations on inherited concepts, functions, and objects. The nomenclature accompanying such an enterprise plays a kind of dialectical role: on the one hand, it pays homage to precedents; on the other hand, it somehow dissolves that to which it pays homage within a greater unificatory crucible, thereby extracting it from the theoretic capsule that had previously contained it. Thus, one both affirms the structural principles of the content of inheritances, whilst negating the enclosures that previously delimited them.

Throughout the PRIMS special issue, one encounters terminology that both recognizes mathematical inheritances and repurposes them within further-abstract theories. For instance, the term "arithmetic holomorphic structure" is invoked periodically, often couched within quotation marks. Some readers may find such an aesthetic choice to be a source of discontentment, as it may be unclear how to avail oneself of quoted, undefined concepts when constructing a Hodge theater. One gathers that their purpose is to lend a kind of scaffolding or commentary to IUT. In fact, much of the IUT text provides a kind of interlacing of object constructions and commentaries thereon. One is reminded of certain novels which employ body text and footnotes in competing portions so as to furnish dual, equipoised currents of narrative and discussion in concert. Nevertheless, given that IUT itself introduces a sizable inventory of new concepts, the presence of further, ancillary concepts has been criticized.

When subject to discussion, however, such ancillary concepts can serve as a meshwork into which the theoretical concepts that attend object constructions can be situated. With this approach in mind, I spent significant time discussing arithmetic holomorphic structure with Mochizuki-sensei.

The first analogy at play in the term arithmetic holomorphic structure is the notion that a ring theory shares a common property with complex-analytic holomorphic structure: they both are constituted by a rigid coupling between underlying dimensions. In the case of complex holomorphy, the dimensions are real and imaginary; in the case of arithmetic holomorphy, the dimensions are those of $\boxtimes$-structure and $\boxplus$-structure.

Mochizuki-sensei:

> Holomorphy means a coupling, a rigid coupling, between two dimensions. So, literally, in the complex case, this rigid coupling gives the notion of a right angle; that's what it refers to. This contrasts with the quasi-conformal situation where right angles are bent. [...] In its most primitive form, [complex] holomorphy refers to what it means to be a square, as opposed to a parallelepiped. [...] A holomorphic map is a homeomorphism that preserves squares. If you start with a topological space, you might consider the category of open subsets. The objects are subsets and the morphisms are inclusions. [...] Basically [...] instead of considering all open subsets, you just consider squares or parallelepipeds.

Thus, holomorphy does not simply refer to a structure that can be decoupled into underlying dimensions, but the structure fused from their rigid coupling.

Mochizuki-sensei:

> It's not just that you have a splitting into two dimensions; the holomorphy lies in this rigid relationship between the two dimensions. You can split a ring into these two combinatorial dimensions, but the holomorphic structure is literally the ring structure.

Nonetheless, decomposition is an operational necessity in IUT, and is thus a practical facet of a Teichmüller theory for arithmetic holomorphic structure. In order to effectuate Teichmüller deformations, one must convert the symmetries (book-kept by certain components of Hodge theaters) into monoid-theoretic data.

Mochizuki-sensei:

> On the one hand, there are these two dimensions, but you can also split it up into local units and local value groups. This decomposition is related to local class field theory.

With the monoid-theoretic data thus obtained, one is in a position to revisit the themes of deformations and Hodge theaters from an arithmetic-holomorphic perspective.

Mochizuki-sensei:

> You're deforming these local value groups, and this is what leads to the Hodge theaters.

The role of quasi-conformal mappings, or Teichmüller deformations, is effectuated by the theta-link.

Mochizuki-sensei:

> Jumping ahead to [...] the theta-link in IUT: [...] you have a situation where the multiplicative structure is preserved, but not the additive structure. [...] The theta-link is not compatible with holomorphic structure.

The log-link, on the other hand, appears as one converts from ring-theoretic to monoid-theoretic data.

Mochizuki-sensei:

> In some sense, the relationship between this [ring] aspect and this [local-class-field-theoretic] aspect is a sort of rotation. In the complex case, you can do rotations for free. It's elementary and often not even mentioned; you can do these rotations. This corresponds to the log-link in IUT. I often mention this.

However, one cannot perform the rotations and deformations together. The incompatibility between them yields a non-commutative log-theta-lattice as an emergent structure.

Mochizuki-sensei:

> You have to temporarily put the rotations on hold. [...] This deformation is not compatible with rotations. [...]

Boyd:

> The fact that you have to put [them] on hold – is this the essence of the non-commutativity of the log-theta-lattice?

Mochizuki-sensei:

> Right, right, right – this is precisely right.

3 The IUT Text

3.1 Form and Content

This section is written to compensate for certain inadequacies that I can already recognize in the last. The preceding historical overview of (certain) genealogical threads is intended not to recapitulate the theory of IUT in full, but to fashion a kind of doorway or portal through which one can enter the textual body. However, it might be the case that the preceding section only offers hints as to what can be found on the other side of the doorway, without fashioning much of a door in its own right. That is to say, one might by now have a sense of the content on the other side of the door, without yet feeling a grip on the handle itself. After all, the doorway to the content of a text is the form of the text itself; one engages the form to access the content. Nonetheless, in the case of IUT, one might possess a summary of the thematic character of the content of the PRIMS special issue whilst still not exactly knowing 'how to begin' reading the volume itself.

I suspect as much particularly in observance of the welter of complaints expressed online over the past few years concerning not merely the content of the PRIMS special issue, but its form. Critics express bewilderment regarding its style of notation; indignation regarding its length and commentary; consternation regarding the plethora of data structures at play; puzzlement as to the relationship between the vastness of IUT and the (perhaps deceptive) succinctness of the *abc* conjecture; and incredulity as to the disparity between its acceptance at RIMS and its controversy on the international stage. Put plainly, it is sometimes implied that the very form of the PRIMS text forestalls engagement with the content.

Whilst reading the IUT papers, one can indeed surmise that there are tacit motivations – and perhaps a kind of implicit thesis on the relations between form and content – guiding the style of the papers. Thus, on certain occasions, Mochizuki-sensei and I did discuss the formal aspects of the PRIMS special issue. What I found is that, indeed, the manner in which the IUT papers are written can very much be viewed as a reification of many of Mochizuki-sensei's mathematical principles. All the more interesting, from my point of view, is the manner in which principles of form and content relate in this respect. For instance, as shall be discussed herein, virtues of canonicality or algorithmicity in mathematics, which Mochizuki-sensei sees as expressed in anabelian geometry, in turn inform the style of the papers. Thus, inquiries into matters of form present an opportunity to further understand Mochizuki-sensei's outlook on mathematics, and the creative respects in which meditations on directions in the mathematical enterprise inspire and guide distinct choices in the production of mathematical works.

Thus, I feel that the foregoing genealogical exercises in the previous section, which suggest how one might navigate the PRIMS text (and prior works) content-wise, might enjoy the supplemental accompaniment of a navigational resource on the textual form of the IUT papers. Thus, I present a section on, in some sense, the 'form' of the IUT texts.

3.2 Algorithmic Constructions

Members of the global arithmetic geometer community have complained that the papers are difficult to read. For instance, the notation itself – the subject of many online memes – is often asserted to be a stylistic element that beleaguers the international readership. It is as though there is a TeX wall sundering lines of communication, a kind of bulwark that cannot be cleared.

In preparation for my interviews with Mochizuki-sensei, I read the PRIMS special issue and the preceding texts. I did so over the course of several years. Progress was very much sub-linear; nonetheless, I tried my best. Such preparations did allow, to my surprise, some degree of substantive discussion on the content of IUT, which elicited the following reaction.

Mochizuki-sensei:

> Your ability to prepare topics that ended up being very natural is again a reflection of your deep reading comprehension. [...] That prompts the fundamental question: why were you able to read the IUT papers, whereas other people were not?

With time, as the two of us discussed the form of the IUT texts, it became rather clear that Mochizuki-sensei possesses a rather different view of the text than notation-frustrated critics. The first hint at this, as unrelated as it might seem, is the frequent appearance of algorithmic themes throughout the text. In the PRIMS special issue, one often encounters terminology such as "reconstruction algorithm", "group-theoretic algorithm", "algorithmic description", "functorial algorithm", etc. Such invocations are far from metaphorical; Mochizuki-sensei views the IUT papers as an algorithmic text. This view informs the notation in the following manner: whereas, perhaps conventionally, notation is used to write down a mathematical computation in the signage of an extant theory, Mochizuki-sensei's notation serves a constructive role closer to that of code – as one might find in, say, a Github repo.

Mochizuki-sensei:

> In the case of IUT, the main driving force for the divergence of writing style is the algorithmic nature of the theory: it's basically one construction. I state this at the beginning of the first IUT paper: it's not a series of papers concerning properties of an already established framework; it's one long construction, essentially.

Algorithmic construction is, in real methodological terms, how Mochizuki-sensei builds the text. Mochizuki-sensei writes the papers using (nested) macros – he doesn't type out the notation manually.

Mochizuki-sensei:

> I try to make the notation as functorial as possible. One sort of creative aspect is the optimization of the use of macros and nested macros. This is one thing that I find strange about students, or young people: [...] they don't like using macros. A macro is a program, essentially. I make all sorts of programs to handle the complicated notation. So, actually, I don't think in terms of the notation that you see in the papers; I think in terms of the macros.

In my case, prior to starting SciSci, my own research was wholly computational. Thus, it isn't my first instinct to think of mathematical content notationally. I think of the content conceptually, with a mapping between nested conceptual relations and the code. After Mochizuki-sensei shared with me his own approach to writing papers, I was struck by the following observation: throughout our conversations on IUT, it is perhaps the case that, for the most part, neither of us had the notation of the IUT papers in mind. Mochizuki-sensei has the macros in mind. In my own case, I suspect that I have my own, coarser, 'reverse-engineered macros' in mind. I know the notation, but I think of it as a kind of culturally contingent residue, a *pro forma* incarnation, of the algorithmic character of the theory. Thus, despite the exquisite chalk collection in the RIMS seminar rooms, we did not spend 16 hours scribbling down all the formulae from the text. We spoke about objects and theoretical structure, which could be denoted using notation or macros.

The algorithmic nature of the text also helps one to understand the role of the extensive commentary included in the papers. Typically, when looking at someone else's code repo, one almost assumes by default that one will not easily make sense of the code; thus, one looks at what is called the "documentation", which is a prosaic description of what the code is and what it does. I interpret the sizable commentary in the PRIMS special issue as comparable to documentation. As I read the IUT texts, I referred to the commentary first, and then looked to see how the commented-upon content was formalized in notation; but I referred to the commentary first. Perhaps, this might partially answer Mochizuki-sensei's question. I didn't stare aghast at the ostensible 'wall of notation' that is so often deemed a source of consternation; I just climbed over the wall to the conceptual side, where the commentary resides, and then continually hopped back and forth between the commentary and the notation, as one would do between documentation and code in a repo.

Here, notation-frustrated critics might protest against this 'macro-view' as perpetuating a kind of author-reader asymmetry, with the former using code that the latter doesn't see and the latter faced with notation that the former doesn't use as a primary reference. However, I'm also not really sure if reading the 'macros behind the paper' would help. On the other hand, given the growing popularity of mathematical formalization, there has been some activity regarding formalization of anabelian geometry with Lean. Mochizuki-sensei, in light of these developments, is often asked about his views on formalization.

Mochizuki-sensei:

> Needless to say, I myself am a complete novice with regard to Lean, but I look forward to seeing to what extent, in the coming years, recent dramatic improvements in Lean-related technologies may be applied to the task of formalizing IUT. The algorithmic nature of IUT should be particularly well-suited to computer formalization by software like Lean.

Thus, to make an optimistic comment, it may very well be the case that the algorithmic approach taken to writing the IUT papers – which, when applied to TeX, has often found poor reception among readers – may lend itself to math formalization.

3.3 Canonicality and the Future of Mathematics

After reading the previous subsection, certain readers might nonetheless ask why Mochizuki-sensei privileges (re-)construction in IUT, at the expense, say, of conforming to a conventional style, and often with the result of lengthy and detailed treatments of certain objects, data, and structures. Attending to such possible questions gives us the occasion to inquire into some of Mochizuki-sensei's deepest-held convictions regarding mathematical virtue and practice.

From conversations with Mochizuki-sensei, I would venture to say that a guiding principle – a kind of lodestar – of his mathematical work is the virtue of canonicality, as practiced by working with canonical objects and developing canonical theories. Most readers will recognize the oft-cited platitude that mathematical results are permanent; once a result is proven, it will serve mathematical posterity in perpetuity. The first motivating gesture that might be given to present Mochizuki-sensei's views on canonicality is to share his suspicion that, in fact, not all mathematical results will enjoy permanence. Rather, indefinite futurity is attained only by canonical work.

Mochizuki-sensei pursues canonicality via detailed construction according to a distinct style, which is often criticized for not conforming to unspoken writing conventions in contemporary mathematical culture. For Mochizuki-sensei, canonicality is an enterprise of detailed construction that resists the trends of culture; therein lies, perhaps, the crux of the academic-cultural aspect of the disagreement on style between Mochizuki-sensei and some arithmetic geometers.

The social and communal affairs of mathematics run the risk, from Mochizuki-sensei's point of view, of allowing the endogenous comforts of a given mathematical milieu to imbue mathematicians with misplaced confidence in the perenniality of their works. That is to say, a mathematical framework may appear to have an indefinite shelf-life, but only because it enjoys the familiarity of current socialization. It may be entirely irrecoverable once the favor of socialization fades as the future unfolds. Such risk is all the more severe, in Mochizuki-sensei's view, in the case of non-canonical approaches. His philosophy on the matter began to develop during adolescence.

Mochizuki-sensei:

> My deep sense [was] that whenever you work with non-canonical structures, everything will be based on a specific setup, or a specific culture, or a specific social context.

In Mochizuki-sensei's assessment, cultural contingencies are more likely to thicken as communities resort to developing mathematical discourses on specific examples. Although seemingly concrete, and perhaps satisfyingly expedient, such discourses may not achieve the generality needed for another (perhaps future) community, for whom all matters of previous cultural contingency and socialization are inaccessible, to avail itself of the past community's work. Mochizuki-sensei sees many such examples in arithmetic geometry.

Mochizuki-sensei:

> Many people, if [they] start looking at an algebraic curve over $\mathbb{C}$, and if they want to understand it, [...] start working with non-canonical structures, non-canonical labels. They consider embeddings into projective space; they start looking at rational functions; they want to deal with concrete polynomials. [...] People justify this approach because they think it's very concrete and it makes them feel that they're really in control of the object, but in some sense you're losing control of the object by doing this.

The test of "control", then, might be the extent to which an outsider can make use of the work for new, unintended ends. Such a prospect might bring to mind the scenario in anabelian geometry where one reconstructs mathematical data (e.g., isomorphism classes of certain hyperbolic curves over number fields) with fidelity from parsimonious input data (e.g., étale fundamental groups). One might then ask if analogous criteria hold for arithmetic geometry more generally; is one's mathematics sufficiently canonical such that a future culture can reconstruct it algorithmically without appeal to one's favorite, culturally tethered examples? Mochizuki-sensei is of the view that non-canonical results might satisfy certain contingent, endogenous desiderata but perish under cultural decoupling, in an exogenous climate. One example might be that of motivic generalizations of Faltings' work.

Mochizuki-sensei:

> There have been certain generalizations of [Faltings'] argument to the case of motives, but [...] it's almost as if you set up some situation where you can do the argument, and then you do the argument. It doesn't give rise to externally interesting results in the same way that Faltings' proof does.

Mochizuki-sensei's concern is that once the enthusiasm for particular examples fades, the apparition of a general mathematical resource will, from the exogenous view of a different mathematical culture, evaporate.

One might attribute the above sensibilities to a kind of anabelian philosophy, but such an attribution does not explain how Mochizuki-sensei came to find anabelian geometry compelling in the first place. Going further into the past, one finds that the exogenous view of mathematical culture expressed above likely owes to the itinerant view on cultural affairs that characterized Mochizuki-sensei's own youth.

Mochizuki-sensei:

> I grew up moving around. I was born in Tokyo and grew up moving back and forth between Japan and the US because my father's company kept transferring him; we kept moving roughly every year or year and a half. So, I was in a situation where it was very difficult to develop normal human relationships; I just concentrated on studying math.

Thus, with the background cultural conditions variable – the invariant being the pillar of mathematics – Mochizuki-sensei began to ponder how one could truly secure an invariant kind of mathematics immune to the vicissitudes of social and cultural fashion.

Mochizuki-sensei:

> I wanted to access a world, I wanted to access objects, that lie beyond specific social or cultural contexts because, no matter how powerful the particular social or cultural context may appear at a certain time, it's always going to be washed away; that's what history shows. That's what my own personal life shows in a very concrete way just by moving around so much.

With this all being said, Mochizuki-sensei is not of the view that one must have such a background in order to appreciate the ephemerality of culture and the prospect of building theories that endure such ephemerality.

Mochizuki-sensei:

> Even if you don't experience that [...] I think it's obvious that things just change so radically. So it's not so much that I was concerned with proving a specific mathematical result. It's just that – I had a deep sense that only things that are truly canonical could survive in the long term. [...] As t goes to infinity, where t is time, I had this deep sense that only truly canonical things would survive.

One can begin to appreciate why one finds novel notation and prodigious construction in the IUT papers. Priority is not given to a theory that is readily metabolizable within the now-exercised channels of mathematical-cultural discourse. Rather, the goal is to build an inter-universal theory in which each Hodge theater knows as much about another Hodge theater in the log-theta-lattice as a future culture knows about us.

One might gather that the reconstructivist goal of the papers is to build a canonical theory in which one can, from the point of view of one Hodge theater (with its collection of Grothendieck universes), recover a ring structure from another Hodge theater that is mutually alien. To treat data structures as mutually alien is to somehow recapitulate, in the present, how a future mathematical culture might relate to our own. Put differently – taking a longitudinal view, mathematical cultures are rather mutually alien; they don't inherit as much from each other as one might think. Thus, mathematical genealogy is, to a degree, a matter of reconstructing data from the inputs one inherits from prior, largely inaccessible, cultures. This is only really viable when the setup is canonical. It is for this reason that the PRIMS special issue is a construction. It may seem 'counter-cultural' with respect to current convention, but this is precisely how mathematical works will read to a future mathematical culture anyway.

Thus, the IUT papers constitute an enterprise in presenting a canonical construction that does not rely on cultural conventions and can, in principle, be accessed by any mathematical culture with uniform facility. Of course, the international mathematical community has experienced some difficulty in interpreting the work, and thus a future culture might also; nonetheless, one gathers that the choice here is to try to distribute the 'probability of recovery' uniformly over future mathematical cultures through canonical construction.

4 Closing Remarks

4.1 On Meeting Mochizuki

I found Mochizuki-sensei to be a kind and patient person. He answered every question that I posed during the interviews, including questions that obviously begged rather belabored answers. (So, for instance, I asked Mochizuki-sensei about the origins of the log-volume approach to Diophantine inequalities during the final interview. Despite the sheer volume of content discussed thereto, his answer – which I don't believe could have been significantly compressed – took about an hour to deliver.) When given questions on matters that he had not considered – such as the relationship between the simulatory roles assigned to prime-strips and species – he found them to be a source of intrigue. He accepted my own mathematical commentary and very much encouraged me to share my own mathematical understanding and views. He was quick to spot errors, which happened a few times. But his doing so, in turn, lent an accentuated sincerity to those occasions on which he assessed that we shared a common understanding of the theory.

I'll conclude with a quote that I think captures Mochizuki-sensei's impression of our exchanges, which says less about me than it does a certain general prospect: a text, no matter how challenging, need not separate human beings – there is always a way to find the author on the other side of the text, and, upon that encounter, to tour the textual surroundings of that meeting spot.

Mochizuki-sensei:

> Another thing I'd like to say is: your background is not in arithmetic geometry. But, you seem to have studied a substantial amount. It's not at the level of a true professional mathematician, but... What is amazing to me is: all of these professional mathematicians – professional arithmetic geometers – say that they tried to study IUT and that they can't make any sense out of it, but you've made a tremendous amount of sense out of it. It's been transmitted to you remarkably well. And that's, of course, to some extent because you are serious about studying it [...] but it serves [as] an existence theorem: it shows [...] that if you really are serious about studying it, there's no reason why you can't study the ideas. [...] I'm just really overjoyed [...] because I've had so much trouble communicating [with] people. There's just all of this noise to the effect of: "it's unreadable; we can't make sense of it", and then here is this person who comes along and makes tremendous sense out of it – just naturally. I haven't really been involved in your study of these ideas. From my point of view: I just have to say that I'm so impressed. So many people have been saying – "it's just impossible, and unapproachable, and inaccessible" – and then someone just comes along and accesses it with such ease. That's really remarkable.

4.2 The Interview Backstory

I have been asked on several occasions how these interviews came to fruition. In short, I owe the opportunity to the generosity of many people.

I first began an email correspondence with Mochizuki-sensei in the autumn of 2023, thanks to an introduction from Professor Ivan FESENKO. I first met Mochizuki-sensei in person in Tokyo in April 2024 at an IUT conference organized by what is now called the ZEN Mathematics Center. I wish to thank Professor KATO Fumiharu (加藤文元) for the invitation, Mr. KAWAKAMI Nobuo (川上量生) for his support of the conference, and Pro-

fessor Fesenko for his recommendation of my invitation. Following the conference, I made a brief trip to RIMS and met with Mochizuki-sensei again, at which point we began to discuss the prospect of further engagements.

In the summer of 2024, I began to discuss the prospect of a SciSci visit to RIMS with Dr. Benjamin COLLAS. Board approval was needed for my visit, but a visit for interviewing purposes is rather unprecedented. Fortunately, Professor FUJIWARA Koji (藤原耕二) at the Kyoto University Mathematics Department, who maintains an active and lively JIR program (数学のジャーナリスト・イン・レジデンス・プログラム) for those interested in conducting interviews with mathematicians at Kyoto University, was willing to accommodate my interviews with Mochizuki-sensei within the program. Furthermore, Dr. Collas, coordinator of the Arithmetic and Homotopic Galois Theory (AHGT) international research network (IRN), was willing to help host me. I thank Dr. Collas and Tamagawa-sensei for their invitation to RIMS and Fujiwara-sensei for allowing me to participate in his program. Typically, visitors to RIMS are professional mathematicians; thus, I required some guidance from Dr. Collas on how to prepare an application in my circumstances. I wish to thank him for his extensive efforts. Fortunately, the RIMS board approved my visit application, and my visit was handled in a routine fashion.

Thus, I wish to thank (in rough chronological order) Professor Fesenko, Kato-sensei, Kawakami-san, Dr. Collas, Tamagawa-sensei, and Fujiwara-sensei. I also wish to thank AHGT, Fujiwara-sensei's JIR program, and RIMS for helping to facilitate the visit.

Background research for the interviews was conducted whilst visiting several research institutes, including the Institute for Pure and Applied Mathematics (IMPA; *Instituto Nacional de Matemática Pura e Aplicada*) in Rio de Janeiro, Brazil; the Nesin Mathematics Village (*Nesin Matematik Köyü*), in Şirince, Türkiye; and the Rotman Institute for Philosophy in London, Ontario, Canada. I'd like to thank these institutions also for their hospitality. I'd also like to thank Jed McCaleb and the Astera Institute for their financial support.

Finally, I wish to thank Mochizuki-sensei for participating in the interviews we scheduled at RIMS; for the many hours he spent with me; for entertaining my unusual methods; and for pardoning my mistakes.

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