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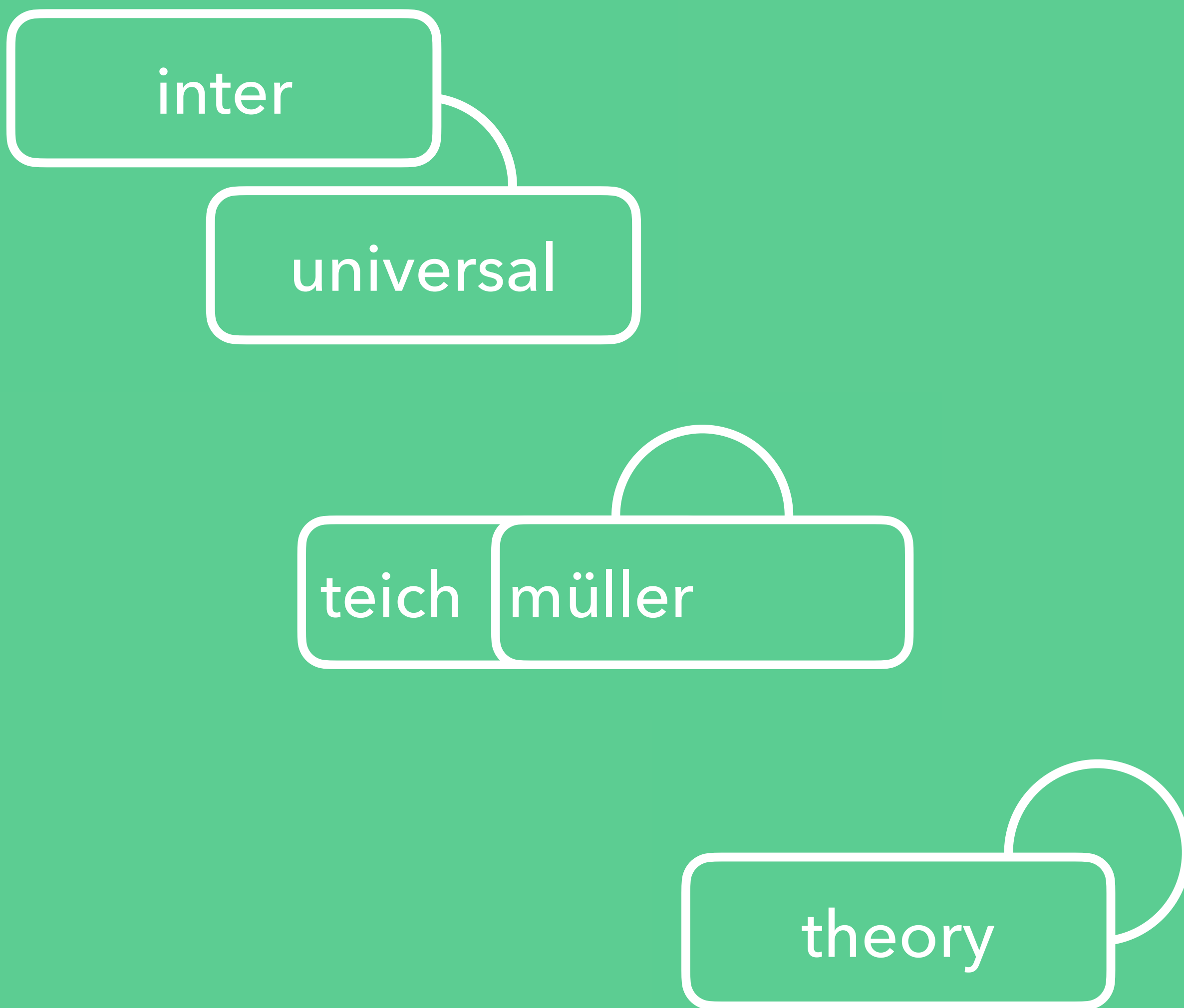

inside the controversy

James Douglas Boyd
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Published by SciSci Research, Inc.

SciSci Research, Inc.
Boulder, Colorado, United States
www.sci-sci.org

Citation Format:

Boyd, J.D. (2025). Inter-universal Teichmüller Theory: Inside the Controversy. WorldSheet Action, 1.

Inter-Universal Teichmüller Theory: Inside the Controversy

James Douglas Boyd

Meeting Mochizuki

SciSci Research just released a report that I wrote on Inter-universal Teichmüller Theory (IUT) based on a series of interviews that I conducted with Professor Mochizuki Shinichi. This is the first time Mochizuki ever agreed to participate in any kind of interview, let alone a series; so, I very much appreciate his participation.

After studying IUT myself for some time, I came to Mochizuki with many technical questions concerning the theory during a month-long visit to the Research Institute for Mathematical Sciences (RIMS), Kyoto University, as part of a collaboration with CNRS (the French National Centre for Scientific Research). Mochizuki answered all questions put to him in detail. With many of the questions pertaining to aspects of IUT that are not made explicit in the four IUT papers published by PRIMS/EMS, we both gained insights from the conversation. After writing the interview report, which communicated Mochizuki's answers to my questions about IUT, I wrote this WorldSheet report to present my own analysis.

Outlook on IUT: Bad and Good

The IUT dispute has indeed been dramatic. As a third party, I'm only really interested in the mathematical stakes of the dispute. I've wanted to understand the dispute and where it can go from here, on mathematical grounds. I am not Mochizuki's political ally, but I wanted to learn about his perspective on IUT, since interaction with Mochizuki is something of a prerequisite for really engaging with the IUT papers. Here, I'll share my own analysis on the dispute and outlook on what I think can happen.

The dispute surrounding IUT is practically frozen by now. Views remain polarized regarding whether or not it proves the *abc* conjecture, with the majority position negative. I'll give my analysis on *abc*, as well as the theory itself, for certain aspects are now being applied beyond *abc*.

Regarding the *abc* conjecture – I don't expect that the IUT proof strategy will find acceptance. I think the concerns raised by the 2018 manuscript by Professor Peter Scholze and Professor Jacob Stix, "Why *abc* is still a conjecture", will remain. Although the manuscript does not use the algorithms that Mochizuki uses in the papers, it does indeed, taking the setup in which the algorithms are applied (i.e., the log-theta-lattice), show that a contradiction can be derived from this setup due to the inter-universal approach that IUT applies to Diophantine inequalities.

In IUT, this log-theta-lattice consists of collections of data and links between them: theta-links and log-links. Otherwise-contradictory relations between data imposed by certain links are offset by assigning data to different universes with distinct labels, with this inter-universal setup giving rise to many copies of $\mathbb{R}$. The Scholze-Stix argument shows that if one simply removes the labels from the beginning (by identifying the various copies of $\mathbb{R}$), and then looks at the theta-links, the contradictions become immediately manifest. Here, I should mention that it's not controversial that IUT uses an inter-universal setup to avoid set-theoretic contradictions (which Mochizuki calls "$\in$-loops"); the Scholze-Stix argument simply shows that this is the case.

So, why has a dispute ensued? The answer is that, although the overall setup clearly harbors these contradictions, suspended by inter-universality, Mochizuki has argued that the papers are misunderstood, for they implement algorithms, he maintains, that can bound heights without triggering any contradiction. This is so, he argues, because, although his algorithms also require eventually removing labels, they do so only much later, following considerable algorithmic work that constitutes the bulk of the proof strategy. Thus, throughout the dispute, Mochizuki has basically been saying that evaluating the papers requires that one remove labels his way (i.e., later) rather than the Scholze-Stix way (i.e., immediately), because theirs yields a contradiction and his doesn't. On the other hand, the Scholze-Stix argument is essentially that it's unnecessary to consider such advanced algorithms since if one just takes the setup and simplifies it down, one finds it to contain a contradiction-inducing mapping protected by labels which one might just as soon remove. It's an extraordinary situation, due to the novelty of using inter-universality to suspend contradictions with labels.

Very few mathematicians have endeavored to look carefully into Mochizuki's algorithms. The debate is being waged on simpler terms; it's about the fact that the contradiction is there in the lattice with the theta-link, whereas Mochizuki's algorithms essentially promise a way around it. It's not about the content of the algorithms, it's about whether mathematicians are confident enough in the basic setup, shown in simplified terms by Scholze-Stix, to even consider the algorithms. I just don't think the *abc* proof strategy will be accepted because it would essentially require the mathematical community to agree to abstain from deriving the contradiction from the setup, despite the ease of doing so, and summoning adequate confidence in the setup to spend years poring over the algorithms. I think they're unlikely to abstain from deriving the contradiction, since mathematicians can remove labels as they so choose; it amounts to a step in one's algorithm, necessary for Diophantine applications. Presenting IUT with the caveat that the setup is sensitive to set-theoretic contradiction and one can only perform a key algorithmic step under highly specific circumstances has been unconvincing.

Before explaining the details, I want to make a sudden turn and, setting the fate of *abc* aside, share a somewhat optimistic point on a different matter. So, I agree that Scholze-Stix are able to make their argument without the algorithms. One might ask if anyone has looked at the math beyond the basics critiqued by Scholze-Stix. The fascinating wrinkle in the story – to talk inside baseball – is that this mathematics is of interest to some mathematicians, and they are engaging with IUT; they mostly come from anabelian geometry and related fields, and – importantly – generally have no interest in *abc*. It's sometimes supposed that these mathematicians must just be cajoled students or *abc* "true believers", but, in fact, much of the interaction is happening in collaboration with CNRS, the largest science funder in Europe.

What happened is twofold. On the one hand, the *abc* proof strategy provoked a global controversy. On the other hand, looking at some of the math in the IUT papers and setting aside *abc*, a relationship between aspects of IUT, anabelian geometry, and related topics such as étale homotopy and Grothendieck-Teichmüller theory did prove attractive to some mathematicians in those areas. In what follows, I'll give my analysis on both developments: why the *abc* proof strategy will likely not be accepted, but why some arithmetic geometers who don't care about *abc* still learn IUT.

"Simulating $\in$-Loops" IUT's Diophantine Strategy

First, we'll start with Scholze-Stix. As a lead-up, I'd like to give a portrait of some of the design choices made for IUT that explicitly target *abc*. I'll often make the point that the structures that comprise the anatomy of IUT come from different sources, some of which can be considered independently of the *abc* issue. One should begin with Diophantine geometry. Let's say one wants to try, in the spirit of Faltings (Mochizuki's PhD advisor), to bound heights for q-parameterized elliptic curves over number fields by showing the heights to be invariant under isogeny. One would like to show that the logarithmic height for q is the same as the logarithmic height for q^l times l. This doesn't work. However, after developing Hodge-Arakelov theory, which concerns evaluation of theta functions at l-torsion points, Mochizuki finds that the argument would work if one could develop a truncated, finite theta-functional method for certain primes of bad reduction. Alas, the problem is that one wants to use only these primes, not all of them. So, Mochizuki builds a "simulation", as he calls it, using these primes alone. This collection of primes constitutes the so-called "prime-strips" in IUT. The "simulation" is essentially a setup in which the prime-strips are treated as equivalent to all primes. The Diophantine goal of IUT is to make this simulation work.

Mochizuki then constructs so-called "Hodge theaters" from prime-strips, each with its own collection of universes. Each Hodge theater is essentially a model of ring/scheme theory. Mochizuki sends q^{j^2}, associated with one Hodge theater, to q, associated with another, via the theta-link. One wants a "gluing" between Hodge theaters such that a prime-strip compatible with both sides can be obtained, thus bounding heights by showing the data from either side of the gluing to agree; that's the simulation. This alone is an untenable method: an equivalence between q^{j^2} and q triggers a cascade of set-theoretic paradoxes, which Mochizuki calls "$\in$-loops". However, one needs these "$\in$-loops", without contradiction.

Aware of these set-theoretic issues, Mochizuki began to address them publicly in the early 2000s as a motivation for inter-universality. As he put it, one needs inter-universality to avoid violating the Axiom of Foundation. The fix was to assign the data to distinct Grothendieck universes and endow them with distinct labels. The prime-strip method is a simulation, and, as written in IUT IV, the "$\in$-loops" are simulated via the use of multiple universes. Simulating "$\in$-loops" amounts to working with differently labeled data assigned to different universes and independently manipulating them to behave as they would in a contradictory relationship without letting the contradiction occur, unless one removes the labels (the crux of the dispute with Scholze-Stix).

However, the theta-link presents another complication: $q^{j^2} \mapsto q$ is non-scheme-theoretic (i.e., not a ring homomorphism); it respects multiplicative structure, but not additive structure, both of which one needs. So, the theta-link threatens to trigger set-theoretic paradoxes, and distorts Hodge theater ring structure. Mochizuki then introduces log-links, which have their origin in p-adic anabelian geometry, to recover the additive structure. Mochizuki then builds a log-theta-lattice, a non-commutative entity consisting of Hodge theaters in the domain and codomain of theta-links and log-links.

Mochizuki saw the distortion by which the theta-link affects the ring structure of Hodge theaters as akin to a Teichmüller dilation (or quasiconformal mapping) in complex Teichmüller theory, which respects real, but not imaginary structure with respect to $\mathbb{C}$. Mochizuki then developed a new Teichmüller theory, in which one dilates the ring/scheme structure with the theta-link, and then tries to reconstruct the additive structure via the log-link, log-invariants, and sophisticated algorithms. Following reconstruction (up to some indeterminacy), one is then to quantify the distortion of the prime-strip simulation and extract a structure common to the Hodge theaters (the "multiradial representation"), and then remove labels.

Evidently, the strategic march of IUT towards *abc* proceeded by a series of maneuvers: one responds to the failure of the isogeny method with a simulation method (e.g., prime-strips, theta-links); one responds to the set-theoretic paradoxes of the simulation method with inter-universality (e.g., different labels, Hodge theaters); and one responds to the non-scheme-theoretic nature of the simulation method by applying a new Teichmüller theory to recover additive structure (i.e., in the log-theta-lattice). Here, the development of "inter-universal Teichmüller theory" can be seen. Thus, the Teichmüller theory and inter-universal framework support the prime-strip-based simulation. It's much easier, in my view, to begin a conversation on Scholze-Stix following this review.

$\mathbb{R}$-Identifications: Scholze and Stix's Critique

The argument of Scholze and Stix is rather simple. They simplify the IUT setup down to two Hodge theaters and consider the codomain and domain of the theta-link between them. As IUT is an inter-universal theory, there are many copies of $\mathbb{R}$ at play. Scholze and Stix maintain that if one is going to bound heights, one must establish some consistency among these copies of $\mathbb{R}$, namely by identifying them. They then go on to demonstrate that if one identifies the many copies of $\mathbb{R}$ at large in IUT, one cannot obtain a nontrivial Diophantine result without a contradiction (i.e., with q^{j^2} and q identified). *Ergo*, the bugs in the prime-strip simulation reveal themselves after one toggles off inter-universality. This is the case, essentially, because if one turns off the $\in$-loop simulation, the prime-strip simulation also fails. This alone should not be controversial; IUT is made inter-universal for this reason.

Mochizuki's argument in response to Scholze-Stix is that they don't use (*inter alia*) the multiradial algorithms or the full log-theta-lattice. Whereas Scholze and Stix remove labels between data, and then take the theta-link, Mochizuki leaves the different labels distinct, and performs transport between certain data in the Hodge theaters in the log-theta-lattice (making extensive use of the log-links), only later to remove the labels. Mochizuki emphasizes that one must consider, at the very least, an "infinite H" in the log-theta-lattice, consisting of two Hodge theaters opposite a theta-link, plus two infinite columns of log-links on either side. One then extracts log-invariants column-wise and pushes them across the theta-link, eventually obtaining a multiradial representation, after which one can remove the labels. This is to be done using anabelian geometry, with abstract groups independent of labeling regimes passed between Hodge theaters. So, the algorithms are intended to extract compatible structure between the Hodge theaters before removing labels.

Mochizuki often refers to the Scholze-Stix viewpoint as the "Redundant Copies School (RCS)" in characterizing their wish to see the copies of $\mathbb{R}$ identified. I myself don't think about the dispute in terms of RCS, and feel that it's unlikely that Scholze and Stix's criticism owes to a general preference to deny distinct copies of mathematical data. The copies identified in the Scholze-Stix manuscript are not any old kind of mathematical data, but copies of $\mathbb{R}$. Thus, I think it might be good to tease out why so many copies of $\mathbb{R}$ might be controversial.

I'd like to address the matter of $\mathbb{R}$-identifications from two different points of view: with respect to Grothendieck universes, and in terms of label-removal. The first is axiomatic, the second algorithmic.

A Question of Universes

With the Scholze-Stix argument simplifying IUT down to a basic aspect of its setup, some might wonder why there is little interest in considering the full theory and Mochizuki's algorithms. One might wonder why they went so far as to describe the *abc* proof strategy as unfixable, and why others concur. I think the answer, in part, is that 1) the theory depends on inter-universality, 2) the general necessity of universes is already debated in mathematics today, and 3) IUT's use of universe-based labeling for suspending contradictions fails under general label-removal. Following their manuscript, I think there's little interest in considering IUT further or whether it is fixable, as the theory strikes many as leading mathematics in an unwanted direction.

A key counterargument to the $\mathbb{R}$-identifications of Scholze-Stix made by Mochizuki is that distinct copies of $\mathbb{R}$ are allowed by the axiom of Grothendieck universes. IUT is founded on ZFCG (i.e., ZFC set theory + the Axiom of Grothendieck universes). ZFC doesn't allow for Grothendieck universes, for Grothendieck universes imply a strongly inaccessible cardinal, which doesn't exist in ZFC. One needs to put Grothendieck universes in by hand; that's why they need an axiom. Since the dispute ensued, Mochizuki has repeatedly emphasized that IUT is merely based on ZFCG, i.e., on SGA (Séminaire de Géométrie Algébrique du Bois Marie), specifically SGA 4. Mochizuki writes in IUT IV that he is not an expert on set-theoretic foundations, but clearly views ZFCG as a natural choice of foundations.

Scholze has consistently taken a position against the use of universes. He has distinguished the κ-condensed sets at play in his condensed mathematics program (developed with Clausen) from the pyknotic sets of Barwick and Haine, with κ-condensed sets enjoying the advantage of not requiring universes. Scholze has asked if Higher Topos Theory can be rewritten without universes. Scholze cites the Stacks Project as emblematic of initiatives that dispense with universes.

McLarty has taken note of Grothendieck's own criticism of universes during his SUNY Buffalo colloquium. Grothendieck also mentions Pierre Samuel's "galaxies", which, as an alternative, constitute sets of sets of rank below a strong limit cardinal. McLarty, then, goes on to liken the category of condensed sets, from the condensed mathematics program of Scholze and Clausen, to galaxies in that the

κ in κ-condensed sets is a strong limit cardinal.

The IUT setup, on the other hand, features a lattice with infinite chains of theta- and log-links of Hodge theaters, each theater with its own collection of Grothendieck universes. Now, some mathematical theories use universes, as they provide a way to, for instance, build category-theoretic constructions (e.g., theories of ∞-topoi) without worrying about cardinality. In IUT, however, universes have a further role: to protect against contradictions. Mochizuki even distinguishes his work from the Grothendieck school in that IUT uses labels for relations that do not respect the labels (which means contradictions ensue without them). IUT isn't employing universes, which are already debated, for the typical use-case. One criticism that could be made of IUT in line with Scholze's positions on universes is that IUT depends on a setup whose contradictions are avoided by fiat, i.e., with an extra axiom, one which Scholze has suggested mathematics avoid. Furthermore, the $\mathbb{R}$-identifications show just how reliant IUT is on the axiom.

Removing Labels

Suppose, nonetheless, that one does allow for ZFCG. Then, because of the way in which IUT assigns labels to data, one must eventually remove the labels. Even in Mochizuki's multiradial algorithms, the labels are eventually removed. Thus, generally speaking, an algorithm for the log-theta-lattice should involve transporting data between Hodge theaters and label-removal. Thus, I would say that Scholze and Stix's approach can be viewed as another, much simpler algorithm: one removes labels first, and then takes the theta-link, which yields a contradiction. Mochizuki has written extensively on the Scholze-Stix approach and why it is not logically related to the algorithms in IUT. In my own view, it's very clear that their approach is not the same; theirs is another algorithm to apply. Their algorithm shows that the necessity of inter-universality is also a vulnerability in the lattice; it's not robust against arbitrary label-removals.

What does one make of a situation in which two parties have two different algorithms for the log-theta-lattice? Let's consider the situation in very basic terms from a computational perspective. A theory is basically just a set of statements. Algorithms run computations that arrive at statements in the theory. A theory is shown to be inconsistent if one can derive a contradiction in it, i.e., if one finds a contradictory statement. One algorithm might avoid those statements, but if one can find another algorithm that reaches a contradiction, one shows the theory to be inconsistent.

So, if IUT is a theory of the log-theta-lattice (in which certain algorithms are then implemented), and if Scholze and Stix have an algorithm that reaches a contradiction, then the theory of the lattice is inconsistent. I think the key issue is that Mochizuki views the algorithms as part of the theory, such that other contradictions are mathematically extrinsic. Nonetheless, I think most mathematicians view the ease with which an immediate contradiction can be derived from the setup as a sign to move on.

What About Lean?

Mochizuki often discusses the IUT papers in algorithmic terms. Few understand IUT, and its *abc* proof strategy is disputed. So, many – including Charles Hoskinson, after whom the Hoskinson Center for Formal Mathematics at Carnegie Melon is named – have suggested that it be formalized in Lean. My own outlook is that Lean won't help in this case, since at issue is this matter of label-removals and $\mathbb{R}$-identifications.

Lean admits distinct type-theoretic universes, which, as Carneiro discusses, if viewed in a set-theoretic framework, are indeed Grothendieck universes. So, on the one hand, I can imagine one trying to formalize the multiradial algorithms using type-theoretic universes with "distinct labeling", perhaps put in by hand. The IUT papers symbolically label the Hodge theaters, q parameters, and other data (e.g., with † or ‡). So, formalizing IUT in a manner consistent with the papers would require encoding labels to prevent data from being identified. One could give them labels, perhaps, with irreducible definitions (or something like that), in order to make them resistant to equivalences. On the other hand, to formalize the Scholze-Stix argument, one would make the data readily amenable to identification. I don't foresee Lean being good for resolving a dispute such as this. Whether or not data is identified or kept distinct is a coding choice, just as it is a symbolic choice in pen-and-paper math. I can imagine both sides finding a way to code up their approach, only to dispute their respective approaches.

"Arithmetic Teichmüller Theory"

Now, I'd like to discuss the source of my, still guarded, optimism regarding the relationship between IUT and anabelian geometry. Scholze and Stix, as they write in their report, set aside a considerable amount of mathematics in the IUT papers in order to simplify the theory down to an aspect of its setup. They don't discuss the reconstruction algorithms in IUT in great detail. These algorithms draw upon notable developments in anabelian geometry (e.g., p-adic anabelian geometry, absolute anabelian geometry, and combinatorial anabelian geometry) that were underway at RIMS at the time and have developed since. Scholze and Stix don't give much attention to the anabelian geometry in IUT, as they remark in their manuscript that they don't see how absolute anabelian geometry, which they suggest is indeed a remarkable development in anabelian geometry, is needed for the IUT proof.

Here, I think the situation is quite nuanced. IUT served as a kind of vehicle for several research areas in anabelian geometry, whose development both preceded and followed the IUT papers. However, IUT employed them with a certain twist – what one might call an "arithmetic Teichmüller theory", or a Teichmüller theory for ring/scheme theory, rather than $\mathbb{C}$ – which is mixed in with the *abc* proof strategy in the IUT papers and never quite spelled out explicitly or distinguished in its own right. This new Teichmüller theory, in a sense, goes beyond what one might expect of absolute anabelian geometry; the Scholze-Stix comment is not so surprising. Before IUT, anabelian geometry was always about the relationship between curves and arithmetic fundamental groups, and absolute anabelian geometry made this relationship quite tight. Arithmetic Teichmüller theory, on the other hand, pertains to relationships between arithmetic fundamental groups under non-scheme-theoretic mappings, which are essentially the "Teichmüller dilations", and anabelian reconstructions. It's essentially a new research program for arithmetic fundamental groups.

Some of the intuition behind arithmetic Teichmüller theory can be seen in the "Absolute Anabelian Geometry III" paper. One can find discussions of dismantling ring structure, non-ring-theoretic mappings and analogies with Frobenius in positive characteristic, Teichmüller dilations and analogies with p-adic Teichmüller theory, etc. However, the paper does not define an arithmetic Teichmüller theory, as a new theory, or explain where it begins and absolute anabelian geometry ends. Rather, it alludes to a "Future 'Teichmüller-like' Extension (?) of Mono-anabelian Theory". This is also the case in IUT too; novel Teichmüller-theoretic concepts such as "arithmetic holomorphic structure" are put in quotes. p-adic Teichmüller theory had its own book; arithmetic Teichmüller theory never got its own treatment. So, the novel anabelian content in IUT, in which one might *a priori* have the most confidence, was overshadowed both due to the proof controversy and the fact that the IUT papers basically jump straight into building Hodge theaters and pursuing the proof without explaining what this new Teichmüller theory really is.

It's difficult to disentangle arithmetic Teichmüller theory from other areas of anabelian geometry, despite it being, in principle, distinguishable. Mochizuki is in the process of parsing out the relationships between topics such as combinatorial anabelian geometry, p-adic anabelian geometry, and IUT; this is the aim of what he is calling the "Interface Papers", which are still being written. Doing so could be helpful, I think.

The other issue, in my view, is that of distinguishing the arithmetic Teichmüller theory from the rest of IUT. I still think one should be able to distinguish the Teichmüller-theoretic aspects of IUT from the *abc* proof strategy and even the inter-universal setup. Mochizuki often says that the multiradial representation in IUT is far more important than *abc*. The multiradial representation is supposed to be a very general result about the multiplicative/additive structure of scheme theory that one learns from arithmetic fundamental groups via the study of non-scheme-theoretic maps and anabelian reconstructions, but if that's the case, it should be amenable to formulation in a manner independent of the setup involving Hodge theaters, prime-strips, and the theta-link. Very few people care about it because of the IUT baggage. I don't see why the main result of arithmetic Teichmüller theory couldn't just be formulated in the familiar terminology of schemes, non-scheme-theoretic morphisms, arithmetic fundamental groups, anabelian reconstruction, and so on, with a short and simple paper.

The theta-link is an example of a non-scheme-theoretic mapping, but the theta-link, prime-strips, and inter-universality are very much embattled because of the $q^{j^2} \mapsto q$ and $\in$-loop issue. On the other hand, the notion of studying arithmetic fundamental groups under non-scheme-theoretic mappings and reconstructions is interesting. However, I doubt many will take an interest if it is instantiated as the multiradial representation, with is attached to the *abc* proof strategy; there must be ways to explain this research area in relatively plain arithmetic-geometric language. I'm hoping that the Interface Papers can distill the vision.

So, in a sense, my source of optimism is my suspicion that there is some interesting and novel anabelian geometry in IUT that for many is almost undetectable in the IUT papers because it is not explicitly communicated and is so closely coupled to the most controversial ingredients of the *abc* proof strategy. Nonetheless, I still remain guardedly optimistic that arithmetic Teichmüller theory can be made plain in étale-homotopic vocabulary. As discussed before, there is interest in IUT among anabelian geometers and étale homotopy researchers who are disinterested in *abc*. Let's explore this development.

The Second Life of IUT

Mochizuki has himself said to me that he's not particularly interested in *abc*. Other mathematicians who have engaged with IUT, such as Assistant Professor Emmanuel Lepage (from the Institut de Mathématiques de Jussieu-Paris Rive Gauche), have, according to Mochizuki, also said that they are not interested in *abc*. Usually, only mathematicians indifferent to *abc* and well-versed in anabelian geometry have walked away satisfied. As also discussed during the roundtable interview, Lepage made a notable advancement in the study of resolution of non-singularities (RNS), a term coined by Professor Tamagawa Akio (an eminent anabelian geometer and Professor at RIMS), and inspired by the work of both Tamagawa and Mochizuki. Mochizuki, from a Teichmüller-theoretic perspective, found a scheme-theoretic interpretation of the use of Kummer and Artin-Schreier coverings in Lepage's work. This has little to do with *abc*; it's an example of a Teichmüller-theoretic intuition finding rather natural purchase in anabelian geometry. Mochizuki and Assistant Professor Tsujimura Shota wrote a preprint on RNS with interesting intersections with the Grothendieck-Teichmüller group. Lepage then gave a lecture series on Berkovich spaces, anabelian geometry, and RNS at RIMS, and a lecture at Sorbonne as part of a workshop organized by Arithmetic and Homotopic Galois Theory (AHGT), a CNRS collaboration (organized by Collas) involving RIMS, École Normale Supérieure, and Université de Lille, which is dedicated to several intertwining topics in étale homotopy, anabelian geometry, and related areas. Notably, both Mochizuki and Stix are also members of the CNRS collaboration, which formally began after the IUT dispute had commenced.

During the SciSci interview, Professor Hoshi Yuichiro (Assistant Professor at RIMS, and also a shining star in anabelian geometry) described some recent advancements made by Tsujimura using cyclotomic synchronization. Hoshi's view was that anabelian geometers can find many useful techniques and concepts in the IUT papers. Indeed, Tsujimura's 2023 paper, published in *Advances in Mathematics*, refers to cyclotomic rigidity, a key algorithmic concept in IUT. There are many similar examples. For instance, a joint 2022 paper by Mochizuki, Hoshi, and Dr. Minamide Arata on Grothendieck-Teichmüller theory and configuration space fundamental groups refers to IUT and absolute anabelian geometry as providing the intuition for constructing GT from an abstract profinite group; this is essentially an arithmetic-Teichmüller-theoretic intuition. This paper (then a preprint), was cited, for instance, by Professor Adam Topaz during a 2021 AHGT Oberwolfach workshop organized by Collas and Stix as well as Professor Nakamura Hiroaki and Professor Pierre Dèbes.

Hoshi is one of the co-organizers of AHGT, the CNRS collaboration. Last year, Tamagawa, Hoshi, Tsujimura, and Dr. Wojciech Porowski (all from the RIMS anabelian geometry community) attended a workshop organized by Stix, Topaz, Professor Anna Cadoret, and Professor Florian Pop at Oberwolfach. Exchange between the global anabelian geometry community and the RIMS anabelian geometry community has been only deepened with the AHGT project. I'm told that young mathematicians who make contact with the CNRS collaboration do often express an interest in IUT (according to CNRS coordinator Benjamin Collas), and the theory is typically discussed through the lens of étale homotopy, a key AHGT theme which pertains to the study of arithmetic fundamental groups, as seen in Ihara's program, Grothendieck's *Esquisse d'un Programme*, and related topics.

Why is it that one can take an interest in IUT from an étale-homotopic perspective without caring about *abc*? So, in the *abc* proof strategy, the theta-link is notorious due to $\in$-loops. One can see its Hodge-Arakelov origins and utilization for the prime-strip-based simulation for *abc*. On the other hand, I think Mochizuki viewed the theta-link as interesting due to it being non-scheme-theoretic. From an anabelian perspective, looking at non-scheme-theoretic mappings is intriguing, for one might see how arithmetic fundamental groups behave, namely by decoupling arithmetic fundamental groups, as groups, from schemes, and finding new functorial relationships. However, one doesn't need to consider the theta-link or Hodge theaters to study this, as evidenced by applications to topics like GT-construction. It should be generalizable, without the IUT baggage; that would make it an arithmetic Teichmüller theory.

This research area, when viewed in its own right, is so simple and curiosity-provoking. As I heard repeatedly that Mochizuki and colleagues aren't particularly interested in *abc*, I thought it such a shame that the mathematics of most interest to them has been swept up in the *abc* proof controversy, one which I doubt will end in Mochizuki's favor. On the other hand, within the high-trust confines of AHGT, which is a very well managed collaborative project, I think there's space being created for setting drama aside and instead forming connections over the anabelian and étale-homotopic core that underlies the arithmetic-Teichmüller-theoretic content that one sees implicitly in the IUT papers.

I still see these forthcoming "Interface Papers", whose contents I do not know, as an opportunity to spell out the étale-homotopic and anabelian core of arithmetic Teichmüller theory, and perhaps really define it as a theory, distinct from absolute anabelian geometry and distinct from IUT. It would be timely, as anabelian geometry at RIMS is ramping up. Mochizuki, Hoshi, and Tsujimura published a fascinating paper in 2025 on a combinatorial construction of Gal $\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$. I've noticed that several of these newer papers don't cite IUT; nonetheless, IUT hasn't gone away. In the background, Mochizuki and colleagues are going after the Section Conjecture, which is the other conjecture put forth in Grothendieck's letter to Faltings. (The other was his anabelian conjecture, which Mochizuki proved in 1996.) The proof strategy involves "new versions of IUT" that pertain to topics such as Grothendieck-Teichmüller theory. I don't yet understand what a "version" of IUT entails. From what I've seen, there are analogies with IUT, which might be articulable simply as analogies with the arithmetic Teichmüller theory in IUT. Lest these future proofs be disregarded in light of the ongoing IUT dispute, I think it will be of the utmost importance to really understand the relationship between arithmetic Teichmüller theory, combinatorial anabelian geometry, absolute anabelian geometry, and étale homotopy. Are they really new versions of IUT, or new applications of this arithmetic Teichmüller theory?

Closing Remarks

So, with the above remarks, I've given my own views on IUT. I've tried to be as direct and plain as possible. Both the SciSci roundtable interview report and the report based on interviews with Mochizuki have served as reference material as I've written these remarks. Those two reports are written in a markedly different style from this analysis. In a manner akin to a documentary or biography, those two reports try to convey the views of Mochizuki and colleagues. Thus, the style of those reports is sympathetic, partly to avoid flavoring the information conveyed with my own views, and partly because my conversations with Mochizuki were very friendly, and I wanted to capture the air of that encounter. However, after finishing the Mochizuki interview report and reflecting on everything I had learned, I wanted to give a relatively cold take on the political situation and the directions in which it can realistically proceed.

Published by SciSci Press